

Quadratic

Passing the arrays to quad so that the points in quad are 1, 2, 3 and 4 is shown in [#poly.for](#) and [#poly.c](#).

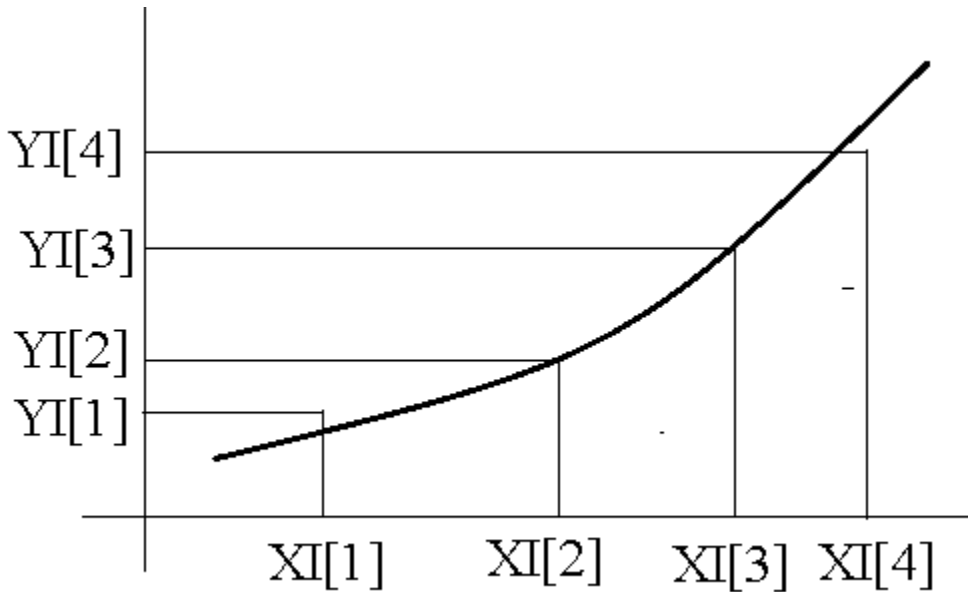


Figure 1 Values YI[1],...,YI[4], at locations XI[1],...,XI[4]

The function described in (1) passes through the points XI[2],YI[2] and XI[3],YI[3], but not through XI[1],YI[1] and XI[4],YI[4]. The first derivatives at the mid points between the data points have errors proportional to the size of the intervals squared (equation (7)). The difference between the first derivatives in the first and last region gives an accurate estimate of the second derivative in the middle region (equation(14)).

Expand the function about $(x_2+x_3)/2$

Expand through third order

$$f(x) = f\left(\frac{x_2+x_3}{2}\right) + \left(x - \frac{x_2+x_3}{2}\right) f'\left(\frac{x_2+x_3}{2}\right) + \frac{1}{2}\left(x - \frac{x_2+x_3}{2}\right)^2 f''\left(\frac{x_2+x_3}{2}\right) + \frac{1}{6}\left(x - \frac{x_2+x_3}{2}\right)^3 f'''\left(\frac{x_2+x_3}{2}\right) \quad (1)$$

Drop the last term and evaluate (1) at x_2 .

$$f(x_2) = f\left(\frac{x_2+x_3}{2}\right) + \left(\frac{x_2-x_3}{2}\right) f'\left(\frac{x_2+x_3}{2}\right) + \frac{1}{2}\left(\frac{x_2-x_3}{2}\right)^2 f''\left(\frac{x_2+x_3}{2}\right) \quad (2)$$

Insert f' from equation(9) and f from equation (18)

$$\begin{aligned}
f(x_2) &= \frac{f(x_2) + f(x_3)}{2} - \frac{1}{2} \left(\frac{x_3 - x_2}{2} \right)^2 f'' \left(\frac{x_2 + x_3}{2} \right) \\
&+ \left(\frac{x_2 - x_3}{2} \right) \left\{ \frac{f(x_3) - f(x_2)}{x_3 - x_2} \right\} \\
&+ \frac{1}{2} \left(\frac{x_2 - x_3}{2} \right)^2 f'' \left(\frac{x_2 + x_3}{2} \right)
\end{aligned} \tag{3}$$

The f'' terms cancel. The equation yields $f(x_2)$ for **all** values of f'' . The same works for x_3 . Thus linear interpolation and quadratic interpolation both yield the correct end values for the region x_2 to x_3 . The object of the next few lines is to provide the best estimate of f'' .

Expand the function about $(x_3+x_4)/2$

Expand x_3 and x_4 about their midpoints.

$$\begin{aligned}
f(x_3) &= f \left(\frac{x_3 + x_4}{2} \right) + \left(x_3 - \frac{x_3 + x_4}{2} \right) f' \left(\frac{x_3 + x_4}{2} \right) + \frac{1}{2} \left(x_3 - \frac{x_3 + x_4}{2} \right)^2 f'' \left(\frac{x_3 + x_4}{2} \right) + \frac{1}{6} \left(x_3 - \frac{x_3 + x_4}{2} \right)^3 f''' \left(\frac{x_3 + x_4}{2} \right) \\
f(x_4) &= f \left(\frac{x_3 + x_4}{2} \right) + \left(x_4 - \frac{x_3 + x_4}{2} \right) f' \left(\frac{x_3 + x_4}{2} \right) + \frac{1}{2} \left(x_4 - \frac{x_3 + x_4}{2} \right)^2 f'' \left(\frac{x_3 + x_4}{2} \right) + \frac{1}{6} \left(x_4 - \frac{x_3 + x_4}{2} \right)^3 f''' \left(\frac{x_3 + x_4}{2} \right)
\end{aligned} \tag{4}$$

Simplify the parenthesis

$$\begin{aligned}
f(x_3) &= f \left(\frac{x_3 + x_4}{2} \right) + \left(\frac{x_3 - x_4}{2} \right) f' \left(\frac{x_3 + x_4}{2} \right) + \frac{1}{2} \left(\frac{x_3 - x_4}{2} \right)^2 f'' \left(\frac{x_3 + x_4}{2} \right) + \frac{1}{6} \left(\frac{x_3 - x_4}{2} \right)^3 f''' \left(\frac{x_3 + x_4}{2} \right) \\
f(x_4) &= f \left(\frac{x_3 + x_4}{2} \right) + \left(\frac{x_4 - x_3}{2} \right) f' \left(\frac{x_3 + x_4}{2} \right) + \frac{1}{2} \left(\frac{x_4 - x_3}{2} \right)^2 f'' \left(\frac{x_3 + x_4}{2} \right) + \frac{1}{6} \left(\frac{x_4 - x_3}{2} \right)^3 f''' \left(\frac{x_3 + x_4}{2} \right)
\end{aligned} \tag{5}$$

The first and third terms in the expansion in line 1 are the negative of the first and third terms on line 2, while the second and fourth are the same. Subtract the first line from the second

$$f(x_4) - f(x_3) = 2 \left(\frac{x_4 - x_3}{2} \right) f' \left(\frac{x_3 + x_4}{2} \right) + \frac{1}{3} \left(\frac{x_4 - x_3}{2} \right)^3 f''' \left(\frac{x_3 + x_4}{2} \right) \tag{6}$$

Or

$$f' \left(\frac{x_3 + x_4}{2} \right) = \frac{f(x_4) - f(x_3)}{x_4 - x_3} - \frac{1}{24} (x_4 - x_3)^2 f''' \left(\frac{x_3 + x_4}{2} \right) \tag{7}$$

The error in (7) on dropping the last term is proportional to $(x_4 - x_3)^2$.

In the same way

$$f' \left(\frac{x_1 + x_2}{2} \right) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{1}{24} (x_2 - x_1)^2 f''' \left(\frac{x_1 + x_2}{2} \right) \tag{8}$$

And

$$f' \left(\frac{x_2 + x_3}{2} \right) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} - \frac{1}{24} (x_3 - x_2)^2 f'' \left(\frac{x_2 + x_3}{2} \right) \quad (9) \leftarrow \text{This is the second term in (1)}$$

Expand the derivative of the function about $(x_2+x_3)/2$

$$f'(x) = f' \left(\frac{x_2 + x_3}{2} \right) + \left(x - \frac{x_2 + x_3}{2} \right) f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(x - \frac{x_2 + x_3}{2} \right)^2 f''' \left(\frac{x_2 + x_3}{2} \right) \quad (10)$$

Evaluate (10) at $(x_1+x_2)/2$ and $(x_3+x_4)/2$

$$\begin{aligned} f' \left(\frac{x_1 + x_2}{2} \right) &= f' \left(\frac{x_2 + x_3}{2} \right) + \left(\frac{x_1 + x_2}{2} - \frac{x_3 + x_2}{2} \right) f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(\frac{x_1 + x_2}{2} - \frac{x_3 + x_2}{2} \right)^2 f''' \left(\frac{x_2 + x_3}{2} \right) \\ f' \left(\frac{x_3 + x_4}{2} \right) &= f' \left(\frac{x_2 + x_3}{2} \right) + \left(\frac{x_3 + x_4}{2} - \frac{x_3 + x_2}{2} \right) f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(\frac{x_3 + x_4}{2} - \frac{x_3 + x_2}{2} \right)^2 f''' \left(\frac{x_2 + x_3}{2} \right) \end{aligned} \quad (11)$$

Simplify the parenthesis

$$\begin{aligned} f' \left(\frac{x_1 + x_2}{2} \right) &= f' \left(\frac{x_2 + x_3}{2} \right) + \left(\frac{x_1 - x_3}{2} \right) f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(\frac{x_1 - x_3}{2} \right)^2 f''' \left(\frac{x_2 + x_3}{2} \right) \\ f' \left(\frac{x_3 + x_4}{2} \right) &= f' \left(\frac{x_2 + x_3}{2} \right) + \left(\frac{x_4 - x_2}{2} \right) f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(\frac{x_4 - x_2}{2} \right)^2 f''' \left(\frac{x_2 + x_3}{2} \right) \end{aligned} \quad (12)$$

Subtract line 1 from line 2

$$f' \left(\frac{x_3 + x_4}{2} \right) - f' \left(\frac{x_1 + x_2}{2} \right) = \frac{1}{2} \left((x_3 + x_4) - (x_1 + x_2) \right) f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{8} \left((x_4 - x_2)^2 - (x_3 - x_1)^2 \right) f''' \left(\frac{x_2 + x_3}{2} \right) \quad (13)$$

Or

$$f'' \left(\frac{x_2 + x_3}{2} \right) = \frac{2 \left(f' \left(\frac{x_3 + x_4}{2} \right) - f' \left(\frac{x_1 + x_2}{2} \right) \right)}{\left((x_3 + x_4) - (x_1 + x_2) \right)} - \frac{1 \left((x_4 - x_2)^2 - (x_3 - x_1)^2 \right)}{4 \left((x_3 + x_4) - (x_1 + x_2) \right)} f''' \left(\frac{x_2 + x_3}{2} \right) \quad (14) \leftarrow$$

Write equation (1) for points 2 and 3

$$\begin{aligned} f(x_2) &= f \left(\frac{x_2 + x_3}{2} \right) + \left(x_2 - \frac{x_3 + x_2}{2} \right) f' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(x_2 - \frac{x_3 + x_2}{2} \right)^2 f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{6} \left(x_2 - \frac{x_3 + x_2}{2} \right)^3 f''' \left(\frac{x_2 + x_3}{2} \right) \\ f(x_3) &= f \left(\frac{x_2 + x_3}{2} \right) + \left(x_3 - \frac{x_3 + x_2}{2} \right) f' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{2} \left(x_3 - \frac{x_3 + x_2}{2} \right)^2 f'' \left(\frac{x_2 + x_3}{2} \right) + \frac{1}{6} \left(x_3 - \frac{x_3 + x_2}{2} \right)^3 f''' \left(\frac{x_2 + x_3}{2} \right) \end{aligned} \quad (15)$$

Simplify the parenthesis

$$\begin{aligned}
 f(x_2) &= f\left(\frac{x_2+x_3}{2}\right) + \left(\frac{x_2-x_3}{2}\right) f'\left(\frac{x_2+x_3}{2}\right) + \frac{1}{2}\left(\frac{x_2-x_3}{2}\right)^2 f''\left(\frac{x_2+x_3}{2}\right) + \frac{1}{6}\left(\frac{x_2-x_3}{2}\right)^3 f'''\left(\frac{x_2+x_3}{2}\right) \\
 f(x_3) &= f\left(\frac{x_2+x_3}{2}\right) + \left(\frac{x_3-x_2}{2}\right) f'\left(\frac{x_2+x_3}{2}\right) + \frac{1}{2}\left(\frac{x_3-x_2}{2}\right)^2 f''\left(\frac{x_2+x_3}{2}\right) + \frac{1}{6}\left(\frac{x_3-x_2}{2}\right)^3 f'''\left(\frac{x_2+x_3}{2}\right)
 \end{aligned} \tag{16}$$

Add the two lines in(16). The f''' term cancels.

$$f(x_2) + f(x_3) = 2f\left(\frac{x_2+x_3}{2}\right) + \left(\frac{x_3-x_2}{2}\right)^2 f''\left(\frac{x_2+x_3}{2}\right) \tag{17}$$

$$f\left(\frac{x_2+x_3}{2}\right) = \frac{f(x_2) + f(x_3)}{2} - \frac{1}{2}\left(\frac{x_3-x_2}{2}\right)^2 f''\left(\frac{x_2+x_3}{2}\right) \tag{18} \leftarrow$$

Assume that the term containing f''' can always be taken to be zero

The two values of f' are given by (7) and (8)

$$f'\left(\frac{x_3+x_4}{2}\right) = \frac{f(x_4) - f(x_3)}{x_4 - x_3} \tag{19}$$

$$f'\left(\frac{x_1+x_2}{2}\right) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{20}$$

Summary

In the region between x_2 and x_3

$$x_m = (x_2 + x_3) / 2$$

$$f(x) = a + b(x - x_m) + c(x - x_m)^2$$

$$f'(x) = b + 2c(x - x_m)$$

$$Fp34 = (dat(4) - dat(3)) / (x(4) - x(3))$$

$$Fp23 = (dat(3) - dat(2)) / (x(3) - x(2))$$

$$Fp12 = (dat(2) - dat(1)) / (x(2) - x(1))$$

$$Fpp23 = 2 * (fp34 - fp12) / ((x_3 + x_4) - (x_1 + x_2))$$

$$F23 = (dat(2) + dat(3)) / 2 - (x_3 - x_2) ** 2 * Fpp23 / 2$$

$$a = F23$$

$$b = Fp12$$

$$c = Fpp23 / 2$$

```

FUNCTION QUAD(XI, YI, X, DQUADDX)
IMPLICIT REAL*8 (A-H, O-Z)
DIMENSION XI(4), YI(4)
B = (YI(3) - YI(2)) / (XI(3) - XI(2))

```

```

FP12=(YI(2)-YI(1))/(XI(2)-XI(1))
FP34=(YI(4)-YI(3))/(XI(4)-XI(3))
C=(FP34-FP12)/(XI(3)+XI(4)-XI(1)-XI(2)) !FPP23/2
A=(YI(2)+YI(3))/2-(XI(3)-XI(2))**2*C/4

```

$$f\left(\frac{x_2 + x_3}{2}\right) = \frac{f(x_2) + f(x_3)}{2} - \frac{1}{2}\left(\frac{x_3 - x_2}{2}\right)^2 f''\left(\frac{x_2 + x_3}{2}\right) \quad (21)$$

```

XM=X-(XI(2)+XI(3))/2
DQUADDX=B+2*XM*C
QUAD=A+XM*(B+XM*C)
RETURN
END

```

```

function Quad(xi,yi:TALAG4;x:double;var dQuaddx:double):double;
var
  a,b,c,xm,fp12,fp34,fpp23:double;
begin
  b:=(yi[3]-yi[2])/(xi[3]-xi[2]);
  fp12:=(yi[2]-yi[1])/(xi[2]-xi[1]);
  fp34:=(yi[4]-yi[3])/(xi[4]-xi[3]);
  c:=(fp34-fp12)/(xi[3]+xi[4]-xi[1]-xi[2]);
  a:=(yi[2]+yi[3])/2-sqr(xi[3]-xi[2])*c/4;
  xm:=(xi[2]+xi[3])/2;
  dQuaddx:=b+2*(x-xm)*c;
  Result:=a+(x-xm)*(b+(x-xm)*c);
end;

```

[tquad.wpj](#) [tquad.zip](#) [ctquad.wpj](#) [ctquad.zip](#)

The data is extended forward and backward so that all points are in the 1, 2, 3, 4 format given above.

[poly.for](#)

```

...
DIMENSION XP(:),DP(:)
COMMON /PASS/IW
SAVE NC,NDAT,XP,DP
DATA NC/0/
IF(NC.EQ.0)THEN
  NC=1
  OPEN(1,FILE='loren.out')
  NDAT=NLINES(1)
  ALLOCATE (XP(0:NDAT+1),DP(0:NDAT+1)) ← this dimensions these to be from 0 to Ndat+1
  DO I=1,NDAT
    READ(1,*)XP(I),DP(I)
  ENDDO
  CLOSE(1)
  XP(0)=XP(1)-(XP(2)-XP(1)) ← this extension could be used in the other interpolation methods
  DP(0)=DP(1) ← this could have included a derivative estimate rather than simple repetition
  XP(NDAT+1)=XP(NDAT)+(XP(NDAT)-XP(NDAT-1))
  DP(NDAT+1)=DP(NDAT)
ENDIF
CALL LOCATE(X,XP,NDAT,J)

```

```
JB=MAX(-1,MIN(J-3,NDAT-3))
```

```
...  
POLY=QUAD(XP(JB+1),DP(JB+1),X,DQUADDX) ← the address of xp(1) in quad is xp(jb+1) in poly.
```

poly.c

```
...  
static double *xa,*ya;  
static int n=0;  
extern int iw;  
int i,jb;  
FILE *fp;  
if(n==0){  
    fp = fopen("loren.out","r");  
    if(fp == NULL){printf(" cannot open file loren.out \n");  
        return 0.0;}  
    n=linecount(fp);  
    xa = (double *) calloc(n+2, sizeof(double)); ← n+2 terms  
    ya = (double *) calloc(n+2, sizeof(double));  
    for (i=1;i<n+1;i++)fscanf(fp, "%lg %lg", &xa[i],&ya[i]);  
    fclose(fp);  
    xa[0]=xa[1]-(xa[2]-xa[1]);  
    ya[0]=ya[1];  
    xa[n+1]=xa[n]+(xa[n]-xa[n-1]);  
    ya[n+1]=ya[n];}  
jb=locate(x,xa,n);  
...  
jb=jb-1; ← I printed and looked a few times to get these lines correctly.  
if(jb > n-2)jb=n-2;  
if(jb < 0 )jb=0;  
...  
fint=quad(&xa[jb-1],&ya[jb-1],x,dquaddx); ← the addresses are pointers that are passed this way.  
return fint;}
```

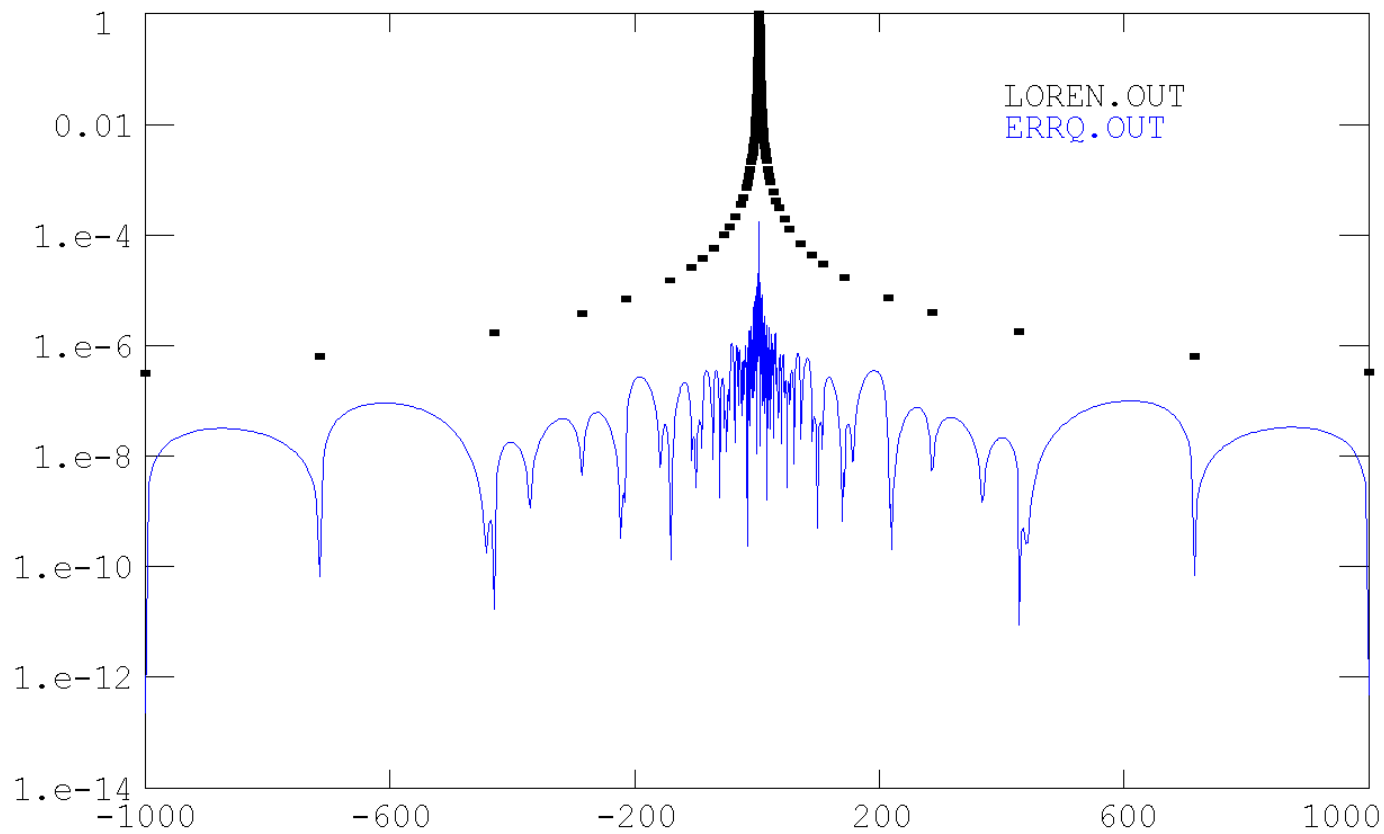


Figure 2 The function and its interpolation error.

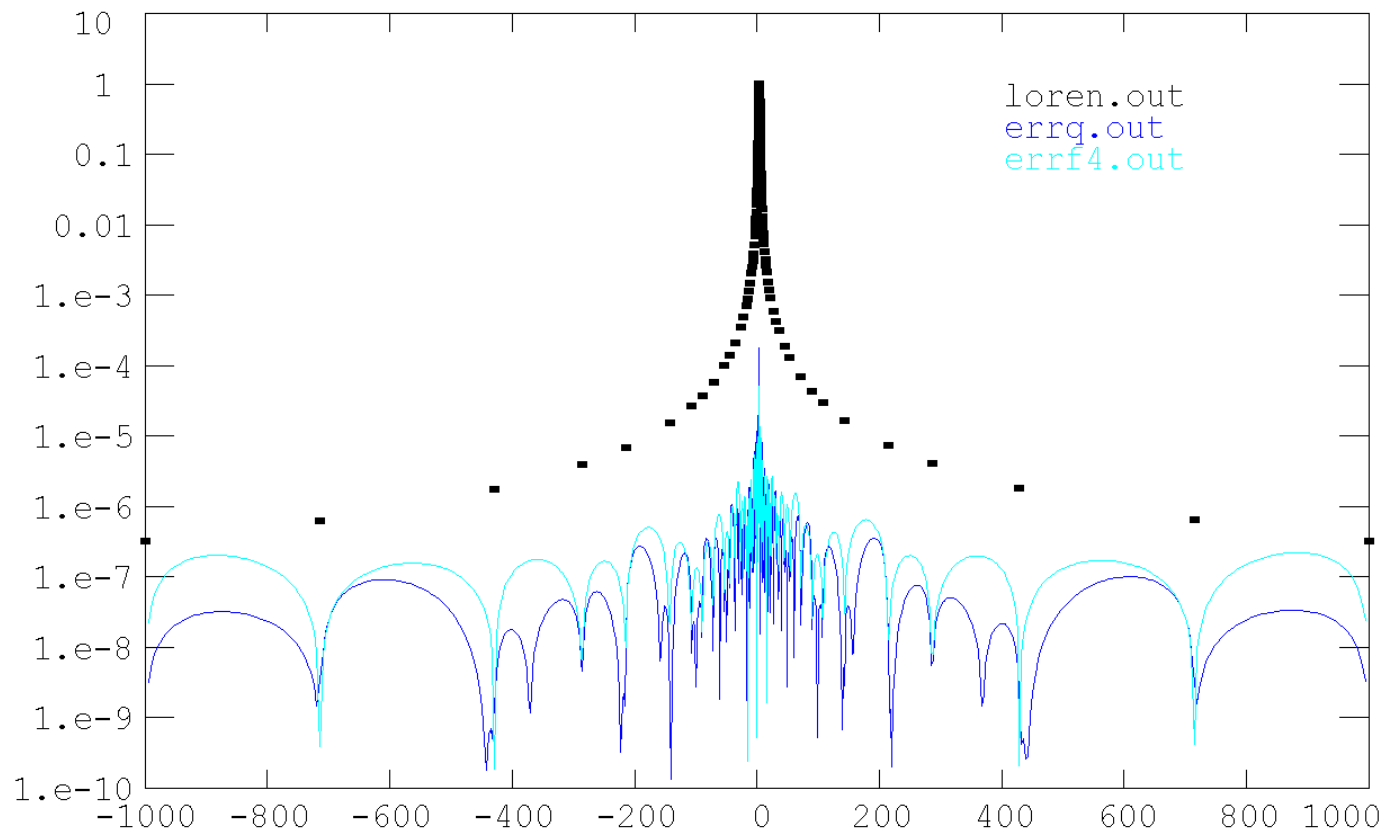


Figure 3 The function (black dots); errq the quadratic error: errf4 is the 4 term Lagrange polynomial error.

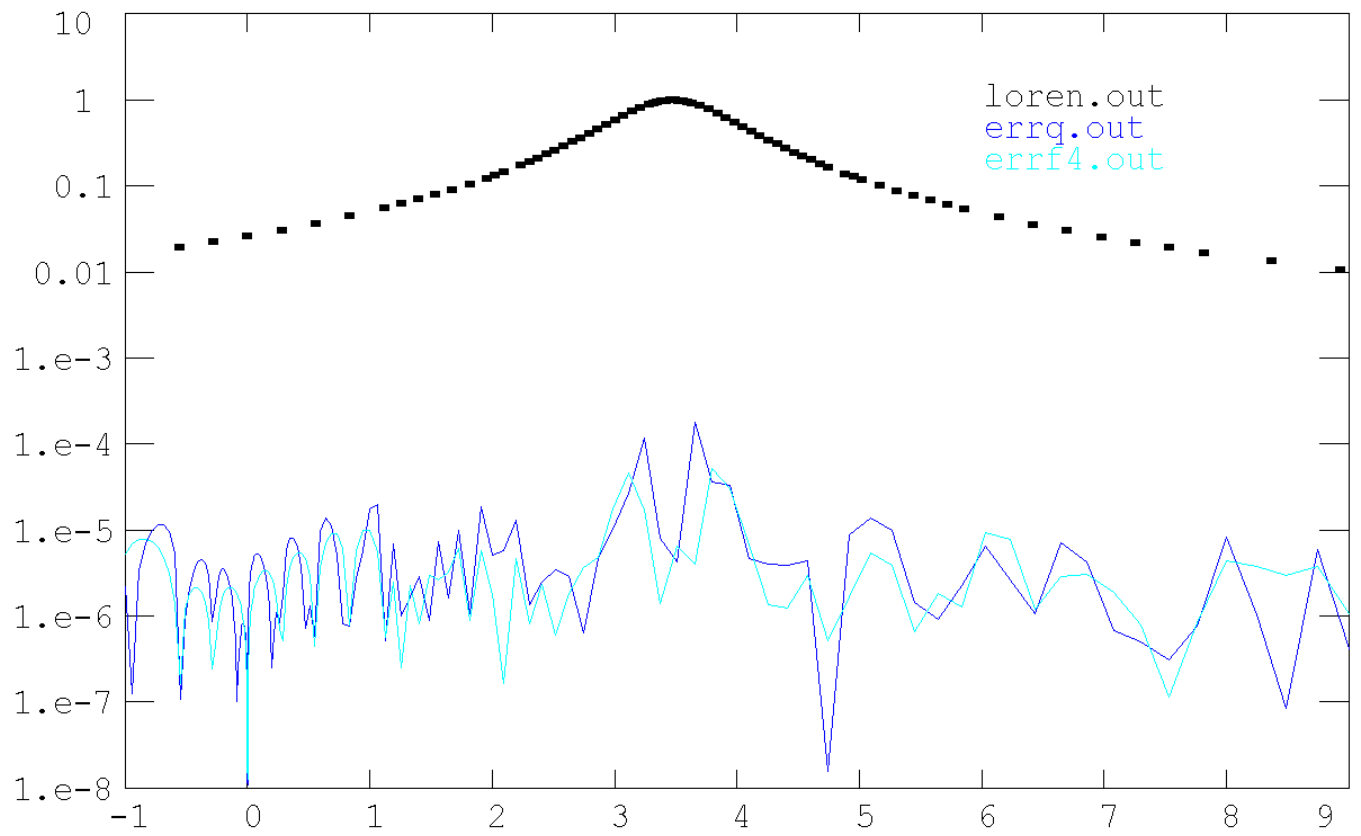


Figure 4 Expansion of the central region. Errf4 has one more term than the quadratic approximation, so it should be better when there are lots of points.