## Summary

The differential equation for the L=1, m=1, state is in (1.12)


## Angular Momentum

$$L = \vec{r} \times \vec{p} \qquad (1.1)$$

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \vec{\varepsilon}_x & \vec{\varepsilon}_y & \vec{\varepsilon}_z \\ x & y & z \\ i\frac{\partial}{\partial x} & i\frac{\partial}{\partial y} & i\frac{\partial}{\partial z} \end{vmatrix} = \vec{\varepsilon}_x \left( iy\frac{\partial}{\partial z} - iz\frac{\partial}{\partial y} \right) + \vec{\varepsilon}_y \left( iz\frac{\partial}{\partial x} - ix\frac{\partial}{\partial z} \right) + \vec{\varepsilon}_z \left( ix\frac{\partial}{\partial y} - iy\frac{\partial}{\partial x} \right) \qquad (1.2)$$

By direct computation

$$(L_x, L_y) = iL_z$$

$$L \times L = \begin{vmatrix} \vec{\varepsilon}_x & \vec{\varepsilon}_y & \vec{\varepsilon}_z \\ L_x & L_y & L_z \\ L_x & L_y & L_z \end{vmatrix} = iL \qquad (1.3)$$

In spherical components [i]

$$\begin{aligned} x &= r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \\ r^2 &= x^2 + y^2 + z^2 \qquad \cos\theta = z/r \qquad \tan\phi = y/x \end{aligned} \qquad (1.4)$$

➔

$$L_x = i\left( \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi} \right) \quad L_y = -i\left( \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi} \right) \quad L_z = -i\frac{\partial}{\partial\theta} \qquad (1.5)$$

$$L^2 = -\left( \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right) \qquad (1.6)$$

The eigenvalues of $L_z$ are

$$\begin{aligned} L_z\psi &= m\psi \\ \psi &= \exp(im\phi) f(r,\theta) \end{aligned} \qquad (1.7)$$

Periodicity in ∅ makes m an integer. Inserting (1.7) into (1.6)

$$-\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{m^2}{\sin^2\theta}\right)f_c^m(r,\theta)=cf_c^m(r,\theta) \quad (1.8)$$

Any component of **L** commutes with $L^2$

Suppose

$$\begin{aligned} L_z f_m &= m f_m \\ (L_x+iL_y)L_z f_m &= m(L_x+iL_y)f_m \end{aligned} \quad (1.9)$$

$$\begin{aligned} (L_x+iL_y)L_z - L_z(L_x+iL_y) &= -(L_x+iL_y) \\ L_z(L_x+iL_y) &= (L_x+iL_y)L_z+(L_x+iL_y) \end{aligned} \quad (1.10)$$

Using (1.10) in (1.9)

$$L_z(L_x+iL_y)f_m=(m+1)(L_x+iL_y)f_m$$

This cannot continue indefinitely. Let $m_1$ be the maximum value m so that $(L_x+iL_y)f_{m1}=0$

$$\begin{aligned} L^2 f_{m1} &= \left(L_x^2+L_y^2+L_z^2\right) \\ &= \left[(L_x-iL_y)(L_x+iL_y)+L_z^2+L_z\right]f_{m1} \\ &= \left(m1^2+m1\right)=m1(m1+1) \end{aligned}$$

With $\boldsymbol{\ell}$ = m1, this becomes

$$L^2 f_\ell = \ell(\ell+1)$$

The raising operator is

$$\begin{aligned} L_x+iL_y &= i\left(\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)+\left(\cos\phi\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \\ &= (\cos\phi+i\sin\phi)\frac{\partial}{\partial\theta}+i\cot\theta(\cos\phi+i\sin\phi)\frac{\partial}{\partial\phi} \qquad (1.11) \\ &= \exp(i\phi)\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right) \end{aligned}$$

Operating on the m=-$\boldsymbol{\ell}$ state this is zero

$$\left(\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)f_l^{\ell}\left(r,\theta\right)\exp\left(i\ell\phi\right)=0$$

$$\frac{\partial}{\partial\theta}f_l^{\ell}\left(r,\theta\right)=\ell\cot\left(\theta\right)f_l^{\ell}\left(r,\theta\right) \qquad (1.12)$$

$$\frac{\partial}{\partial\theta}\ln f_l^{\ell}\left(r,\theta\right)=\ell\cot\left(\theta\right)$$

$$\ln f_l^{\ell}\left(r,\theta\right)=\ell\ln\left(\sin\theta\right)$$

So that

$$Y_\ell^{\ell}\left(\theta,\phi\right)=\sin^{\ell}\theta\exp\left(i\ell\phi\right) \qquad (1.13)$$

$$\begin{aligned}L_x-iL_y&=i\left(\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)-\left(\cos\phi\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\\&=-\left(\cos\phi+i\sin\phi\right)\frac{\partial}{\partial\theta}+i\cot\theta\left(\cos\phi+i\sin\phi\right)\frac{\partial}{\partial\phi} \qquad (1.14)\\&=-\exp\left(i\phi\right)\left(\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\phi}\right)\end{aligned}$$

Operating on the state m=-$\ell$

$$\left(\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\phi}\right)f_l^{\ell}\left(r,\theta\right)\exp\left(-i\ell\phi\right)=0$$

$$\frac{\partial}{\partial\theta}f_l^{\ell}\left(r,\theta\right)=-\ell\cot\left(\theta\right)f_l^{\ell}\left(r,\theta\right) \qquad (1.15)$$

$$\frac{\partial}{\partial\theta}\ln f_l^{\ell}\left(r,\theta\right)=\ell\cot\left(\theta\right)=\ell\tan\left(\theta\right)$$

$$\ln f_l^{\ell}\left(r,\theta\right)=\ell\ln\left(\sin\theta\right)$$

So that

$$Y_\ell^{-\ell}\left(\theta,\phi\right)=\sin^{\ell}\theta\exp\left(-i\ell\phi\right) \qquad (1.16)$$

Not surprisingly this is the same as $Y_\ell$.

The operator[ii]

$$L^2 = \left(\vec{r}\times\vec{p}\right)\bullet\left(\vec{r}\times\vec{p}\right) = r^2\nabla^2 + \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \quad (1.17)$$

So that

$$-\nabla^2\Psi\left(\vec{r}\right) = \frac{\ell\left(\ell+1\right)}{r^2} - \frac{2}{r}\frac{\partial\Psi}{\partial r} - \frac{\partial^2\Psi}{\partial r^2} \quad (1.18)$$

The second term is a bit of a surprise, making the Schrodinger equation in the limit that r→0

$$\frac{\ell\left(\ell+1\right)}{r^2}\Psi\left(r\right) - \frac{2}{r}\frac{\partial\Psi}{\partial r} - \frac{\partial^2\Psi}{\partial r^2} + \left(V\left(r\right)-E\right)\Psi\left(r\right) = 0 \quad (1.19)$$

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[i] D. Bohm, **Quantum Theory,** Prentice Hall (1951) pp 313-317

[ii] For example see E. Merzbacher, **Quantum Mechanics,** John Wiley (1961) section 9.2