Notes from phz 6607, Special and General Relativity
University of Florida, Fall 2004, Detweiler

Exact Solutions of the Einstein Equations

These notes are not a substitute in any manner for class lectures. Please let me know if you find errors.

I. MINKOWSKI SPACE

The metric of Minkowski space is

$$g_{ab}dx^a dx^b = -dt^2 + dx^2 + dy^2 + dz^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{1}$$

where

$$\begin{aligned} x &= r\sin\theta\,\cos\phi \\ y &= r\sin\theta\,\sin\phi \\ z &= r\cos\theta; \end{aligned} \tag{2}$$

t and r have ranges

$$-\infty < t < \infty \quad \text{and} \quad r \geq 0. \tag{3}$$

This same geometry looks rather different in "double null" coordinates,

$$\begin{aligned} u &= t - r \\ v &= t + r \end{aligned} \tag{4}$$

or

$$\begin{aligned} r &= \frac{1}{2}(v-u) \\ t &= \frac{1}{2}(v+u), \end{aligned} \tag{5}$$

where the u and v have ranges

$$-\infty < u + v < \infty \quad \text{and} \quad v - u > 0 \tag{6}$$

and are called "null" coordinates because $g^{ab}\nabla_a u\ \nabla_b u = 0 = g^{ab}\nabla_a v\ \nabla_b v$. A sketch of Minkowski space in double null coordinates is in Fig. 2. In these double null coordinates

$$g_{ab}dx^a dx^b = -du dv + \frac{1}{4}(v-u)^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{7}$$

Note that even though this metric might look odd, regularity is determined by the invertibility of the metric, and this metric is perfectly regular everywhere except when $v - u = 0$, which corresponds to $r = 0$.

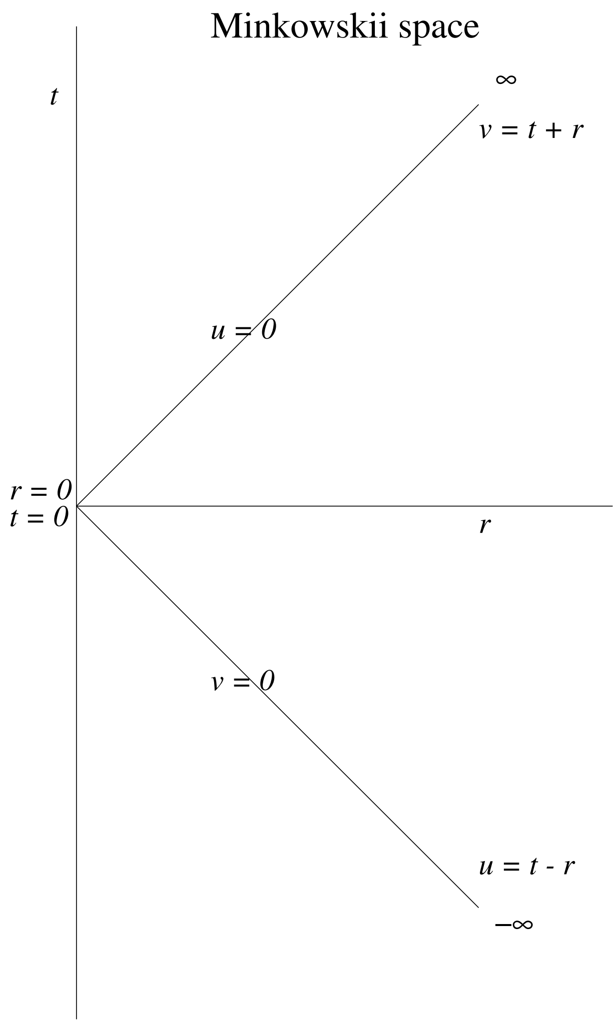

FIG. 1: Minkowski space in double null coordinates. t and r axes are vertical and horizontal, respectively, with ranges $-\infty < t < \infty$ and $0 \leq r < \infty$. The null coordinate axes are on 45° lines and are bounded by $-\infty < u < \infty$ and $-\infty < v < \infty$ and the additional constraint that $v - u > 0$ which implies that $r > 0$. Note that below the u axis $v < 0$, and below the v axis u is negative.

For reasons which might become clear later, it is useful to define coordinates p and q from

$$\begin{aligned} \tan p &= u \\ \tan q &= v \end{aligned} \tag{8}$$

with ranges

$$-\frac{\pi}{2} < p < \frac{\pi}{2}, \quad -\frac{\pi}{2} < q < \frac{\pi}{2} \quad \text{and} \quad q \geq p \tag{9}$$

so that

$$\begin{aligned} v - u &= \tan q - \tan p \\ &= (\sin q \ \cos p - \sin p \ \cos q)/\cos q \ \cos p \\ &= \sin(q - p) \ \sec p \ \sec q. \end{aligned} \tag{10}$$

Also,
$$\sec^2 p \, dp = du \quad \text{and} \quad \sec^2 q \, dq = dv. \tag{11}$$

Now in p, q coordinates Minkowski space is

$$g_{ab} dx^a dx^b = \frac{1}{4} \sec^2 p \ \sec^2 q \left[-4dp \, dq + \sin^2(q-p) \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] \tag{12}$$

Now, the part of the metric in square brackets is similar to the metric for Minkowski space in Eq. (7). So define a new tensor

$$\bar{g}_{ab} dx^a \, dx^b = -4dp \, dq + \sin^2(q-p) \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{13}$$

which is interesting in its own right. The two metrics g_{ab} and $\bar{g}_{ab}$ are an example of "conformally" related metrics where
$$\bar{g}_{ab} = \Omega^2 g_{ab} \tag{14}$$
for some scalar field Ω.

Define new coordinates T and R by

$$T = q + p \quad \text{and} \quad R = q - p \tag{15}$$

and
$$\bar{g}_{ab} dx^a \, dx^b = -dT^2 + dR^2 + \sin^2 R \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{16}$$

When the coordinates T and R have ranges

$$-\infty < T < \infty \quad \text{and} \quad 0 < R < \pi \tag{17}$$

$\bar{g}_{ab}$ is the metric of the "Einstein static universe," a homogeneous, isotropic, time independent solution of the Einstein equations, with a cosmological constant and perfect fluid. The three-volume of each $T =$ constant surface, is a three-sphere.

For the Minkowski space that we started with the range of the coordinates is

$$-\pi < T < \pi \quad \text{and} \quad 0 < R < \pi. \tag{18}$$

Thus a simple, finite picture of the entire Minkowski space can be drawn on the blackboard using this conformal transformation.

A summary of the coordinate relationships is

$$\begin{aligned} t &= \frac{\sin T}{\cos R + \cos T}, \\ r &= \frac{\sin R}{\cos T + \cos R}. \end{aligned} \tag{19}$$

A sketch of the geometry is in Fig. 2.

What is notable is that the radial null geodesics of both Minkowski space and the conformally related $\bar{g}_{ab}$ geometry move along 45° lines. This is the first example of what is loosely referred to as a "Penrose diagram." Particular parts of the "embedding of Minkowski space" within an extended Minkowski space are labeled in the diagram. Specifically,

- i_+, future timelike infinity, is "located at" $t = \infty$ for any fixed value of r

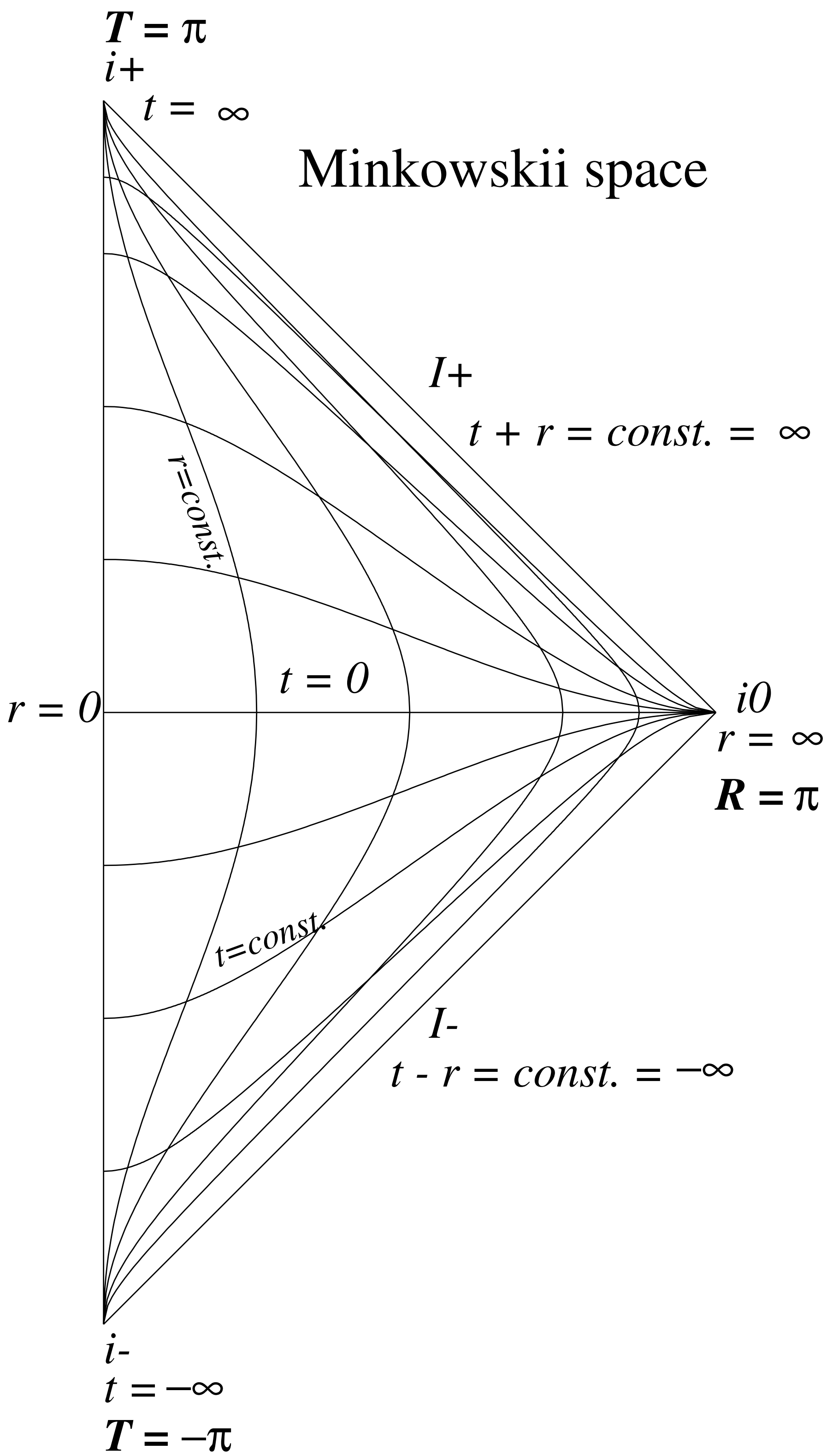

FIG. 2: Minkowski space in a Penrose diagram. T and R axes are vertical and horizontal, respectively, with ranges $-\pi < T < \pi$ and $0 \leq R < \pi$. Null geodesics are at 45° from the axes. Note that the bisector of the angle at the intersection of a $t =$ constant and $r =$ constant line are also at 45° from the axes.

- i_-, past timelike infinity, is "located at" $t = -\infty$ for any fixed value of r
- i_0, spatial infinity, is "located at" $r = \infty$ for any fixed value of t
- $\mathcal{I}^+$, pronounced "scrI-plus" or future null infinity, is "located at" $r + t = \infty$ for any fixed value of $r - t$. This is where all null geodesic go.
- $\mathcal{I}^+$, pronounced "scrI-minus" or past null infinity, is "located at" $r - t = \infty$ for any fixed value of $r + t$. This is where all null geodesic come from.

The understanding of Penrose diagrams is, perhaps, the single most important tool for developing physical intuition in general relativity.

A. An aside on conformal transformations

The two metrics g_{ab} and $\bar{g}_{ab}$ which are related by

$$\bar{g}_{ab} = \Omega^2 g_{ab} \tag{20}$$

for some scalar field Ω are said to be "conformally equivalent" or "conformally related."

The Christoffel symbols of these two metrics are related by

$$\bar{\Gamma}^a_{bc} = \Gamma^a_{bc} + \Omega^{-1}\left(\delta^a_b \partial_c \Omega + \delta^a_c \partial_b \Omega - g_{bc} g^{ad} \partial_d \Omega\right). \tag{21}$$

The relationship between the Riemann tensors, Ricci tensors and scalar curvatures for two conformally related metrics may be derived in a straightforward, if tedious, manner from Eq. (21). One result is that

$$\bar{R} = \Omega^{-2} R - 2(n-1)\Omega^{-3} g^{cd} \nabla_c \nabla_d \Omega - (n-1)(n-4)\Omega^{-4} g^{cd} \nabla_c \Omega \, \nabla_d \Omega \tag{22}$$

for an n dimensional manifold; see Hawking and Ellis p 42.

A useful and interesting feature of conformally related metrics on the same manifold are that null geodesics for one metric are also null geodesics for the other metric. This is easy to see: first, a null vector for g_{ab} is trivially a null vector of $\bar{g}_{ab}$. And second, let k^a be the tangent vector to a null geodesic of g_{ab}, then

$$\begin{aligned} k^b \bar{\nabla}_b k^a &= k^b \nabla_b k^a + k^b \Omega^{-1}\left(\delta^a_b \partial_c \Omega + \delta^a_c \partial_b \Omega - g_{bc} g^{ad} \partial_d \Omega\right) k^c \\ &= k^b \nabla_b k^a + 2k^a k^b \partial_b \ln \Omega = 2k^a k^b \partial_b \ln \Omega \end{aligned} \tag{23}$$

where the first equality follows from the relationship between the Christoffel symbols, the second follows from the fact that k^a is null, and the third from k^a being a geodesic of g_{ab}. However, $k^b \bar{\nabla}_b k^a 2k^a k^b \partial_b \ln \Omega$ implies that k^a is, indeed tangent to a null geodesic, and the fact that the right hand side here is not zero only implies that the parameter along the geodesic is not an affine parameter.

Note: Only null geodesics are preserved under a conformal transformation—all other geodesics are not necessarily invariant under a conformal transformation.

Note: To say that a metric is conformally flat is to say nothing directly about the actual curvature of the metric. For example, in the next subsection it is proven that all two dimensional metrics are conformally flat. In particular this implies that the usual surface of a two-sphere is conformally flat; however, it is quite clear that the two-sphere has plenty of curvature.

Note: It is easy to see that the angle between two vectors, which are not null, is preserved under a conformal transformation from

$$\frac{A^a B^b g_{ab}}{(A^a A^b g_{ab})^{1/2}(B^a B^b g_{ab})^{1/2}} = \frac{A^a B^b \bar{g}_{ab}}{(A^a A^b \bar{g}_{ab})^{1/2}(B^a B^b \bar{g}_{ab})^{1/2}}, \tag{24}$$

and also that a pair of orthogonal vectors remain orthogonal under a conformal transformation.

B. All two dimensional geometries are conformally flat

For any metric on a two dimensional manifold, a particular coordinate system exists where the metric is conformally flat, i.e. the metric is the product of a scalar field (the conformal factor χ below), with a flat metric. The components of the metric in one coordinate system (x, y) is related to metric in a second coordinate system (ϕ, ψ) by

$$\begin{aligned} g_{ab}dx^a dx^b &= g_{xx}dx^2 + 2g_{xy}dxdy + g_{yy}dy^2 \\ &= \chi(d\psi^2 \pm d\phi^2) \end{aligned} \tag{25}$$

where the transformation properties of the components of a tensor imply that

$$g^{AB} = g^{ab}\frac{\partial X^A}{\partial x^a}\frac{\partial X^B}{\partial x^b}. \tag{26}$$

Thus, we look for functions $\psi(x, y)$ and $\phi(x, y)$ such that

$$g^{\psi\phi} = g^{xx}\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial x} + g^{xy}\left(\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x}\right) + g^{yy}\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial y} = 0, \tag{27}$$

and

$$\begin{aligned} g^{\psi\psi} \mp g^{\phi\phi} &= g^{xx}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 \mp \left(\frac{\partial\phi}{\partial x}\right)^2\right] + 2g^{xy}\left[\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} \mp \frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}\right] \\ &\quad + g^{yy}\left[\left(\frac{\partial\psi}{\partial y}\right)^2 \mp \left(\frac{\partial\phi}{\partial y}\right)^2\right] = 0. \end{aligned} \tag{28}$$

While being tedious, direct substitution shows that Eq. (27) is satisfied if

$$\frac{\partial\phi}{\partial x} = \kappa\left(g^{xy}\frac{\partial\psi}{\partial x} + g^{yy}\frac{\partial\psi}{\partial y}\right) \tag{29}$$

and

$$\frac{\partial\phi}{\partial y} = -\kappa\left(g^{xx}\frac{\partial\psi}{\partial x} + g^{xy}\frac{\partial\psi}{\partial y}\right) \tag{30}$$

for any arbitrary function κ:

$$\begin{aligned}
g^{\psi\phi} &= g^{xx}\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial x} + g^{xy}\left(\frac{\partial\psi}{\partial x}\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial x}\right) + g^{yy}\frac{\partial\psi}{\partial y}\frac{\partial\phi}{\partial y} \\
&= \left(g^{xx}\frac{\partial\psi}{\partial x} + g^{xy}\frac{\partial\psi}{\partial y}\right)\frac{\partial\phi}{\partial x} + \left(g^{xy}\frac{\partial\psi}{\partial x} + g^{yy}\frac{\partial\psi}{\partial y}\right)\frac{\partial\phi}{\partial y} \\
&= \kappa\left(g^{xx}\frac{\partial\psi}{\partial x} + g^{xy}\frac{\partial\psi}{\partial y}\right)\left(g^{xy}\frac{\partial\psi}{\partial x} + g^{yy}\frac{\partial\psi}{\partial y}\right) \\
&\qquad - \kappa\left(g^{xy}\frac{\partial\psi}{\partial x} + g^{yy}\frac{\partial\psi}{\partial y}\right)\left(g^{xx}\frac{\partial\psi}{\partial x} + g^{xy}\frac{\partial\psi}{\partial y}\right) \\
&= 0
\end{aligned} \tag{31}$$

Substitution of Eqs. (29) and (30) into Eq. (28) yields

$$\begin{aligned}
g^{\psi\psi} \mp g^{\phi\phi} &= g^{xx}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 \mp \left(\frac{\partial\phi}{\partial x}\right)^2\right] + g^{xy}\left[\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial y}\frac{\partial\phi}{\partial x}\right] + g^{yy}\left[\left(\frac{\partial\psi}{\partial y}\right)^2 \mp \left(\frac{\partial\phi}{\partial y}\right)^2\right] \\
&= g^{xx}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 \mp \kappa^2\left(g^{xy}\frac{\partial\psi}{\partial x} + g^{yy}\frac{\partial\psi}{\partial y}\right)^2\right] \\
&\quad + 2g^{xy}\left[\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} \pm \kappa^2\left(g^{xx}\frac{\partial\psi}{\partial x} + g^{xy}\frac{\partial\psi}{\partial y}\right)\left(g^{xy}\frac{\partial\psi}{\partial x} + g^{yy}\frac{\partial\psi}{\partial y}\right)\right] \\
&\quad + g^{yy}\left[\left(\frac{\partial\psi}{\partial y}\right)^2 \mp \kappa^2\left(g^{xx}\frac{\partial\psi}{\partial x} + g^{xy}\frac{\partial\psi}{\partial y}\right)^2\right] \\
&= \left[g^{xx}\left(\frac{\partial\psi}{\partial x}\right)^2 + 2g^{xy}\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y} + g^{yy}\left(\frac{\partial\psi}{\partial y}\right)^2\right]\left\{1 \mp \kappa^2\left[g^{xx}g^{yy} - (g^{xy})^2\right]\right\}.
\end{aligned} \tag{32}$$

Thus, to satisfy Eq. (28) we require that

$$\pm\kappa^2 = \left[g^{xx}g^{yy} - (g^{xy})^2\right]^{-1} = \det g. \tag{33}$$

From now on assume that $\sqrt{g}$ means $\sqrt{|g|}$. Now Eqs. (29) and (30) are

$$\partial_x\phi = \sqrt{g}g^{ya}\partial_a\psi \tag{34}$$

and

$$\partial_y\phi = -\sqrt{g}g^{xa}\partial_a\psi. \tag{35}$$

These have the integrability condition

$$\partial_y\left(\sqrt{g}g^{ya}\partial_a\psi\right) = -\partial_x\left(\sqrt{g}g^{xa}\partial_a\psi\right) \tag{36}$$

or

$$\nabla^a\nabla_a\psi = 0. \tag{37}$$

Thus any solution of Laplace's equation on the two surface, in the original coordinates, provides a coordinate transformation to a metric of the form given above in Eq. (25) with the conformal factor χ given by $\chi^{-1} = (\nabla^a\psi)\nabla_a\psi$.

II. SCHWARZSCHILD GEOMETRY

The Schwarzschild geometry is

$$g_{ab}dx^a dx^b = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{38}$$

where the range of t is

$$-\infty < t < \infty \quad , \tag{39}$$

and the range of r is expected to be

$$r > 0; \tag{40}$$

however, at this point it appears as though there are difficulties extending the r coordinate through the value $2m$ where this form of the metric becomes singular.

A. Radial null geodesics

Light cones govern causality in any spacetime. Thus, the null geodesics are always of physical interest.

We can introduce outgoing and ingoing null coordinates u and v which are similar to those used for Minkowski space,

$$u = t - r_* \quad \text{and} \quad v = t + r_* \tag{41}$$

where

$$r_* = r + 2m \ln\left|\frac{r}{2m} - 1\right| \tag{42}$$

so that

$$dr_* = \frac{dr}{1-2m/r}. \tag{43}$$

In terms of v and r coordinates, the Schwarzschild geometry is

$$g_{ab}dx^a dx^b = -\left(1-\frac{2m}{r}\right)dv^2 + 2dvdr + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2. \tag{44}$$

The covariant form of this metric is

$$g^{ab}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b} = +2\frac{\partial}{\partial v}\frac{\partial}{\partial r} - \left(1-\frac{2m}{r}\right)\left(\frac{\partial}{\partial r}\right)^2 + r^2\left(\frac{\partial}{\partial\theta}\right)^2 + r^2\sin^2\theta\left(\frac{\partial}{\partial\phi}\right)^2; \tag{45}$$

note that g^{ab} exists for $-\infty < v < \infty$ and $0 < r < \infty$. With this form of the metric, nothing particularly interesting happens when $r = 2m$.

And the t Killing vector is $t^a\partial/\partial x^a = \partial/\partial v$ in (v, r, θ, ϕ) coordinates (Check that this is so). For a radial geodesic with four-velocity $u^a = (dv/d\tau, dr/d\tau)$, a conserved quantity E is $t^a u^b g_{ab} \equiv -E$; in other words

$$E = (1-2m/r)\frac{dv}{d\tau} - \frac{dr}{d\tau}. \tag{46}$$

And from the normalization of the four-velocity

$$-(1-2m/r)\left(\frac{dv}{d\tau}\right)^2 + 2\frac{dv}{d\tau}\frac{dr}{d\tau} = [-1, 0] \tag{47}$$

where the right hand side is -1 for timelike geodesics and 0 for null geodesics.

Thus an “ingoing” null geodesic has $dr/d\tau = -E < 0$ and $v =$ const.

B. Radial timelike geodesics

The existence of the timelike Killing vector $t^a \partial/\partial x^a = \partial/\partial t$ implies that a geodesic with four velocity u^a has a conserved quantity $t_a u^a \equiv -E = -(1-2m/r)dt/d\tau$ thus, the components of the tangent vector of a radial geodesic are

$$u^a = \left(\frac{E}{1-2m/r}, \frac{dr}{d\tau}, 0, 0 \right). \tag{48}$$

The constancy of the normalization of u^a provides a first integral of the geodesic equation,

$$-1 = u^a u_a = -\frac{E^2}{1-2m/r} + \frac{(dr/d\tau)^2}{1-2m/r} \tag{49}$$

or

$$\frac{dr}{d\tau} = \pm\sqrt{E^2 - 1 + \frac{2m}{r}}. \tag{50}$$

The $\pm$ determines whether the geodesic is ingoing or outgoing. The case $E = 1$ corresponds to free fall from infinity, if moving inward, or having just enough energy to reach infinity if moving outward. This case is easily integrated

$$\tau_2 - \tau_1 = \pm \int_{r_1}^{r_2} \sqrt{\frac{r}{2m}} dr = \pm\frac{4m}{3} \left[\left(\frac{r_2}{2m}\right)^{3/2} - \left(\frac{r_1}{2m}\right)^{3/2} \right] \tag{51}$$

to find the proper time that it takes for an object, falling from infinity, to move from r_1 to r_2. Let $r_2 = 2m$, and this equation gives the finite proper time to fall from r_1 to $2m$. Similarly, an object with $E = 1$ moving outwards from $r = 0$ reaches $r = 2m$ in a finite amount of proper time, and also reaches any other finite radius r_2 in a finite amount of proper time.

If a geodesic is initially moving outwards then Eq. (50) implies that the turning point occurs at

$$r_{\max} = \frac{2m}{1-E^2} \tag{52}$$

which is at $r = 2m$ if $E = 0$ and is always outside $r = 2m$ for any other value of E. This is strange behavior: any geodesics moving outward and beginning inside $r = 2m$ will always at least reach $r = 2m$ in a finite amount of proper time, and if $E < 1$ will subsequently fall back to $r = 0$, again in a finite amount of proper time.

The behavior of the t coordinate along some radial geodesics is a bit problematic. It is clear that

$$\frac{dt}{d\tau} = \frac{E}{1-2m/r}. \tag{53}$$

If $E > 0$ then $dt/d\tau \to \infty$ as $r \to 2m$. But r reaches $2m$ in a finite amount of proper time, and t becomes infinite logarithmically in $r - 2m$. Once the geodesic has moved to where $r < 2m$ then t decreases as proper time increases.

It is even possible to have a geodesic with $E = 0$: r starts at 0, grows to $2m$ and decreases back to 0 in a finite amount of proper time; for the entire trip t is constant! We must conclude that the coordinate t bears little resemblance to what we normally think of as time when $r \lesssim 2m$.

The previous few paragraphs are more confusing than they need to be. The secret is to focus on the null geodesics: these put restrictions on how timelike world lines can behave.

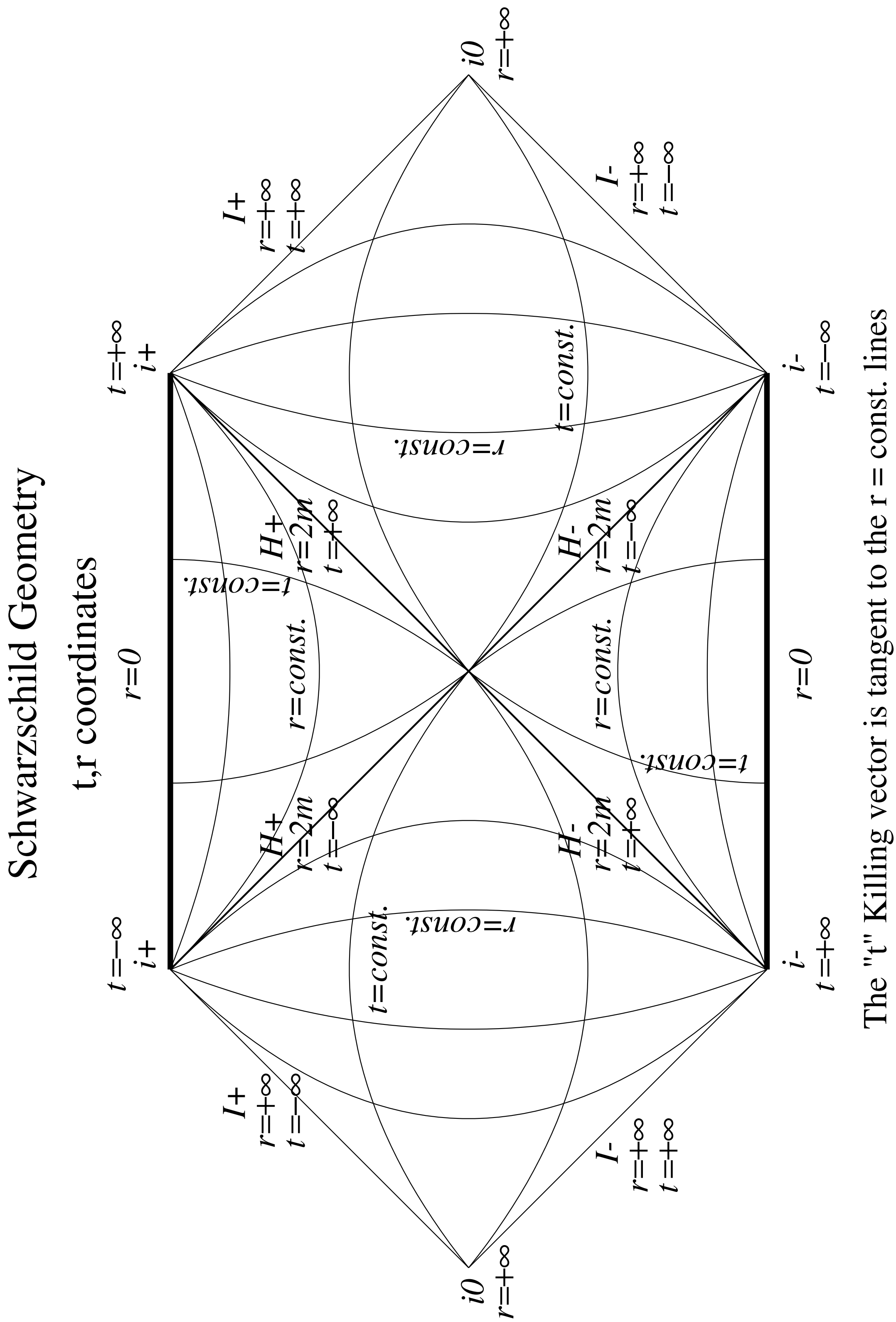

FIG. 3: A Penrose diagram of the maximal analytic extension of the Schwarzschild geometry with labels for the usual Schwarzschild coordinates, t and r.

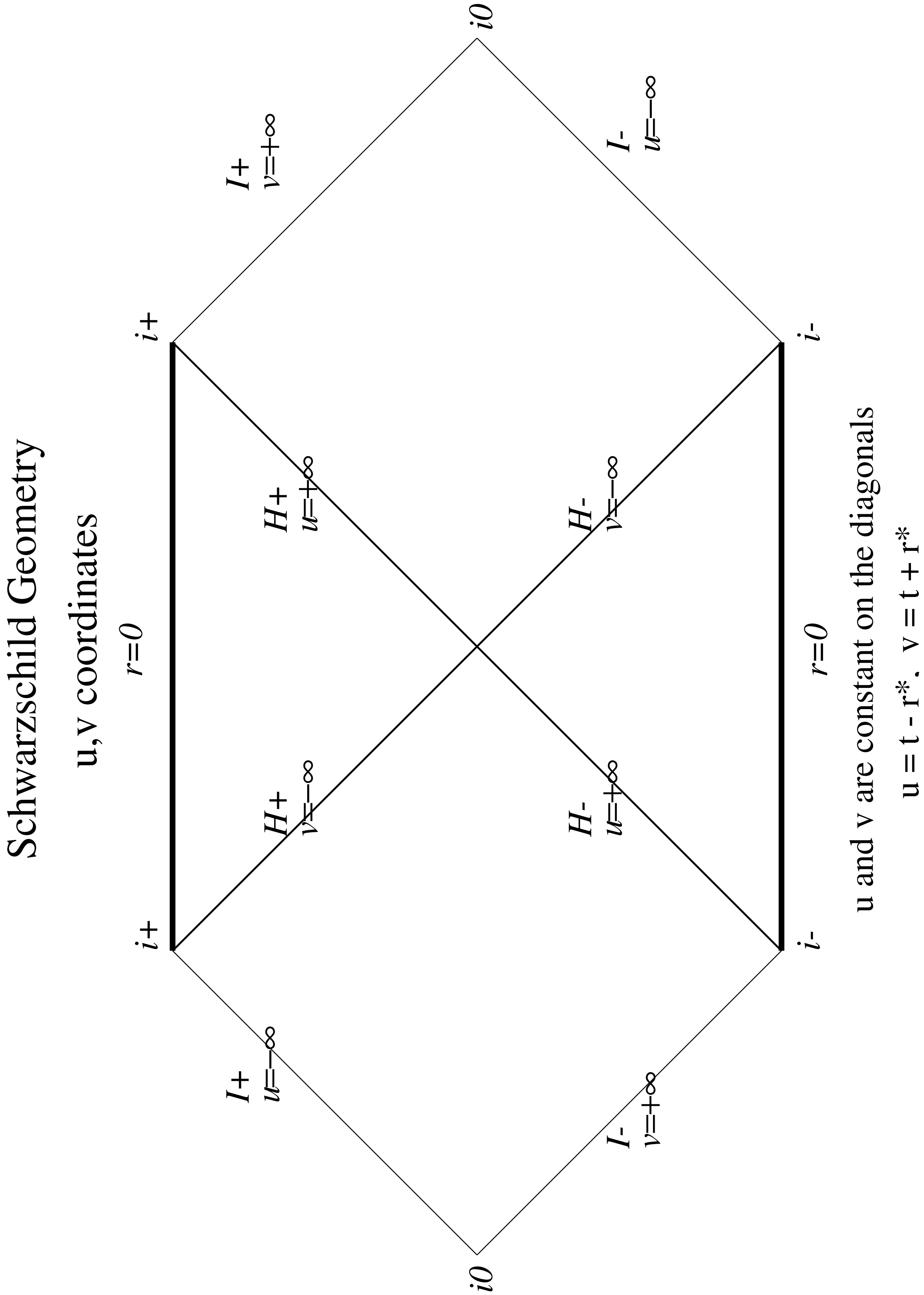

FIG. 4: A Penrose diagram of the maximal analytic extension of the Schwarzschild geometry with labels for the u and v null coordinates.

C. Schwarzschild in isotropic coordinates

Find a spatial coordinate transformation that takes the Schwarzschild metric in Eq. (38) to a form where the spatial part of the metric is conformally flat. In other words, we are looking for a new coordinate system (t, ρ, Θ, Φ) where

$$\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) = \psi^4[d\rho^2 + \rho^2(d\Theta^2 + \sin^2\Theta d\Phi^2)] \tag{54}$$

for some conformal factor ψ. The geometry on the left is spherically symmetric, and it makes sense to preserve this symmetry. Thus, let $\Theta = \theta$, $\Phi = \phi$ and $\rho = \rho(r)$. From the angular parts, we must have

$$r = \psi^2 \rho, \tag{55}$$

and from the radial part of the metric

$$\left(1 - \frac{2m}{r}\right)^{-1/2} dr = \psi^2 d\rho. \tag{56}$$

After a substitution from Eq. (55) for ψ, this becomes

$$(1 - 2m/r)^{-1/2} dr/r = d\rho/\rho. \tag{57}$$

Integration of both sides gives

$$\ln(r - m + \sqrt{r^2 - 2mr}) = c_1 + \ln\rho, \tag{58}$$

or

$$r - m + \sqrt{r^2 - 2mr} = c_2\rho, \tag{59}$$

and let $c_2 = 2$ so that $r \approx \rho$ at large r. Solve this for r to obtain

$$r = \rho(1 + m/\rho + m^2/4\rho^2) = \rho(1 + m/2\rho)^2. \tag{60}$$

We can substitute this into Eq. (55) and we find that

$$\psi = 1 + m/2\rho. \tag{61}$$

Note that

$$\rho = m/2 \quad \text{when} \quad r = 2m \tag{62}$$

locates the event horizon, and that

$$1 - \frac{2m}{r} = \frac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2}. \tag{63}$$

In these new coordinates the Schwarzschild geometry is

$$-ds^2 = -\frac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2} dt^2 + (1 + m/2\rho)^4 (d\rho^2 + \rho^2\, d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2) \tag{64}$$

or

$$-ds^2 = -\frac{(1 - m/2\rho)^2}{(1 + m/2\rho)^2} dt^2 + (1 + m/2\rho)^4 (dx^2 + dy^2 + dz^2) \tag{65}$$

where $\rho^2 = x^2 + y^2 + z^2$ and the "usual" expressions relate θ and ϕ to x, y and z, where ρ (rather than r) plays the role of the radius. These last two expressions for g_{ab} are known as the "isotropic form" of the Schwarzschild geometry.

For some enlightenment, let $\rho \Rightarrow 4m^2/\rho$. This inverts the geometry through the sphere at $\rho = m/2$, but the functional form of the metric is left unchanged. This shows that the region of spacetime where $\rho \to 0$ is geometrically similar to that at $\rho \to \infty$. In other words, $\rho \to 0$ is a "different" asymptotically flat region of the universe, and the sphere at $\rho = m/2$ truly represents a wormhole.

Try to figure out where this other asymptotically flat region is located in the original Schwarzschild coordinates of Eq. (38).