Notes from phz 7608, Special and General Relativity
University of Florida, Spring 2005, Detweiler

Notes on perturbation methods in general relativity

These notes are not a substitute in any manner for class lectures. Please let me know if you find errors.

Each of sections V to XII discuss some aspect of perturbation theory, or derive some useful relationship. However, each section essentially stands alone; I might not actually discuss each of these sections in class.

I. FIRST ORDER PERTURBATION ANALYSIS

Perturbation analysis provides the framework for an understanding of the effects of a small mass moving through a "background" spacetime.

The analysis begins with a background spacetime metric g_{ab} which is usually a vacuum solution of the Einstein equations $G_{ab}(g) = 0$. An object of small mass μ then disturbs the geometry by an amount $h_{ab} = \mathrm{O}(\mu)$ which is governed by the perturbed Einstein equations with the stress-energy tensor $\delta T_{ab} = \mathrm{O}(\mu)$ of the object being the source,

$$E_{ab}(h) = -8\pi\delta T_{ab} + \mathrm{O}(\mu^2). \tag{1}$$

Here $E_{ab}(h)$ is the linear, second order differential operator on symmetric, two-indexed tensors schematically defined by

$$E_{ab}(h) \equiv -\frac{\delta G_{ab}}{\delta g_{cd}} h_{cd}, \tag{2}$$

and G_{ab} is the Einstein tensor of g_{ab}, so that

$$\begin{aligned} 2E_{ab}(h) \;=\; & \nabla^2 h_{ab} + \nabla_a\nabla_b h - 2\nabla_{(a}\nabla^c h_{b)c} \\ & + 2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c\nabla^d h_{cd} - \nabla^2 h) \\ & - 2R^c_{(a} h_{b)c} + R h_{ab} - g_{ab} R^{cd} h_{cd}. \end{aligned} \tag{3}$$

with $h \equiv h_{ab} g^{ab}$ and ∇_a and $R_a{}^c{}_b{}^d$ being the derivative operator and Riemann tensor of g_{ab}. In terms of $\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2} g_{ab} h^c{}_c$, this same equation is

$$\begin{aligned} 2E_{ab}(h) \;=\; & \nabla^2 \bar{h}_{ab} - 2\nabla_{(a}\nabla^c \bar{h}_{b)c} \\ & + 2R_a{}^c{}_b{}^d \bar{h}_{cd} + g_{ab}\nabla^c\nabla^d \bar{h}_{cd} \\ & - 2R^c_{(a}\bar{h}_{b)c} + R\bar{h}_{ab} - g_{ab}R^{cd}\bar{h}_{cd} \end{aligned} \tag{4}$$

(see MTW [13]).

If h_{ab} is a solution of Eq. (1) then it follows from Eq. (2) that $g_{ab} + h_{ab}$ is an approximate solution of the Einstein equations with source $T_{ab} + \delta T_{ab}$,

$$G_{ab}(g + h) = 8\pi(T_{ab} + \delta T_{ab}) + \mathrm{O}(\mu^2). \tag{5}$$

The Bianchi identity implies that

$$\nabla^a E_{ab}(h) = 0 \tag{6}$$

for any symmetric tensor h_{ab}; this is discussed in section VI. Thus, an integrability condition for Eq. (1) is that the stress-energy tensor T_{ab} be conserved in the background geometry g_{ab},

$$\nabla^a \delta T_{ab} = \mathrm{O}(\mu^2). \tag{7}$$

Perturbation analysis at the second order is no more difficult formally than at the first. But the integrability condition for the second order equations is that T_{ab} be conserved not in the background geometry, but in the first order perturbed geometry. Thus, before solving the second order equations, it is necessary to change the stress-energy tensor in a way which is dependent upon the first order metric perturbations. This modification to T_{ab} is said to result from the "self-force" on the object from its own gravitational field and includes the dissipative effects of what is often referred to as "radiation reaction" as well as other nonlinear aspects of general relativity. This modification to T_{ab} is $\mathrm{O}(\mu^2)$ because T_{ab} itself is $\mathrm{O}(\mu)$.

A description of general, nth order perturbation analysis is given in section VII. The procedure is similar to that just outlined. The stress-energy tensor must be conserved with the metric $g^{(n-1)}_{ab}$ in order to solve the nth order perturbed Einstein equation (60) for $h^{(n)}_{ab}$. In an implementation, the task then alternates between solving the equations of motion for the stress-energy tensor and solving the perturbed Einstein equation for the metric perturbation. Similar alternation of focus between the equations of motion and the field equations is present in post-Newtonian analyses.

For many interesting situations the object is much smaller than the length scale of the geometry through which it moves. We expect, then, that the detailed structure of the source should be unimportant in determining its subsequent motion.

To focus on those details of the self-force which are independent of the object's structure we first attempt to model the object by an abstract point particle with no spin angular momentum or internal structure. The stress-energy tensor of a point particle is

$$\delta T^{ab} = \mu \int_{-\infty}^{\infty} \frac{u^a u^b}{\sqrt{-g}} \delta^4(x^a - X^a(s))\, \mathrm{d}s \tag{8}$$

where $X^a(s)$ describes the world line Γ of the particle in some coordinate system as a function of the proper time s along the world line.

The naive replacement of a small object by a delta-function distribution for the stress-energy tensor is satisfactory at first order in the perturbation analysis. The integrability condition Eq. (7) requires the conservation of the perturbing stress-energy tensor. For a point particle this implies that the world line Γ of the particle is an approximate geodesic of the background metric g_{ab}, with $u^a \nabla_a u^b = \mathrm{O}(\mu)$ (*cf* section VIII). The solution of Eq. (1) is formally straightforward, even for a distribution valued source. This procedure has been used many times to study the emission of gravitational waves by a point mass orbiting a black hole [1–3].

II. VECTOR AND TENSOR HARMONICS

Many interesting problems involve perturbations of a spherically symmetric spacetime; these include perturbations of the Minkowskii metric, the Schwarzschild metric and also metrics for non-rotating neutron stars.

We can take advantage of a spherically symmetric background by decomposing various scalar, vector and tensor fields in terms of scalar, vector and tensor spherical harmonics.

For the angular components of vectors and tensors, we find it convenient to follow Thorne's description of the pure-spin vector and tensor harmonics [4], which are closely related to the harmonic decomposition used by Regge and Wheeler [1]. For example, the spin-1 vector harmonics generated by the spherical harmonic function $Y_{\ell m}$ are the even parity

$$Y_a^{\mathrm{E}\ell m} = r\sigma_a{}^b \nabla_b Y_{\ell m} \tag{9}$$

and the odd parity

$$Y_a^{\mathrm{B}\ell m} = -r\epsilon_a{}^b \nabla_b Y_{\ell m}, \tag{10}$$

where

$$\sigma_{ab} \equiv g_{ab} + u_a u_b - n_a n_b \tag{11}$$

is the metric of a constant t,r two-sphere, and

$$\epsilon_{ab} \equiv -\epsilon_{trab} \tag{12}$$

is the Levi-Civita tensor on the same two-sphere. Here

$$\epsilon_{tr\theta\phi} = r^2 \sin\theta, \tag{13}$$

also

$$\epsilon_{\theta\phi} = -r^2 \sin\theta, \tag{14}$$

and u_a and n_a are the unit normals of surfaces of constant t and constant r, respectively.

In the usual Schwarzschild coordinates the components of $\sigma_A{}^B$ and $\epsilon_A{}^B$ are

$$\sigma_A{}^B = \delta_A^\theta \delta_\theta^B + \delta_A^\phi \delta_\phi^B \tag{15}$$

and

$$\epsilon_A{}^B = -\delta_A^\theta \delta_\phi^B (\sin\theta)^{-1} + \delta_A^\phi \delta_\theta^B \sin\theta. \tag{16}$$

In the same coordinates the components of u_a and n_a are

$$u_a = -(1 - 2M/r)^{1/2} \delta_a^t \tag{17}$$

$$n_a = (1 - 2M/r)^{-1/2} \delta_a^r. \tag{18}$$

The terms "even" and "odd" parity are often replaced by "electric" and "magnetic," or by "polar" and "axial."

We generalize this approach: For a vector field ξ_a, the parts $\sigma_a{}^b \xi_b$ which are tangent to a two-sphere may be described by two "potentials" ξ^{ev} and ξ^{od} via

$$\sigma_a{}^b \xi_b = r\sigma_a{}^b \nabla_b \xi^{\mathrm{ev}} - r\epsilon_a{}^b \nabla_b \xi^{\mathrm{od}}. \tag{19}$$

The potentials ξ^{ev} and ξ^{od} are generally functions of all of the spacetime coordinates and are guaranteed to exist by the invertibility of the two dimensional Laplacian on a two-sphere. The factors of r are included for convenience.

The notation for a covariant vector field is condensed by defining even and odd parity vectors associated with the potential $\xi^{\rm ev}$

$$\xi^{\rm ev}_a \equiv r\sigma_a{}^b\nabla_b\xi^{\rm ev} \tag{20}$$

and with the potential $\xi^{\rm od}$

$$\xi^{\rm od}_a \equiv -r\epsilon_a{}^b\nabla_b\xi^{\rm od}. \tag{21}$$

The four independent components of a covariant vector in a spherically symmetric geometry may be written as a sum of the form

$$\xi_a\,{\rm d}x^a = \xi_t\,{\rm d}t + \xi_r\,{\rm d}r + \left(\xi^{\rm ev}_A + \xi^{\rm od}_A\right)\,{\rm d}x^A \tag{22}$$

in terms of the four functions ξ_t, ξ_r, $\xi^{\rm ev}$ and $\xi^{\rm od}$. The capital index A is used here just as a reminder that the vector to which it is attached is tangent to the two-sphere. The A index should otherwise be considered an ordinary spacetime index in the covariant spirit of Eq. (19)-Eq. (21).

Similarly for a symmetric tensor field h_{ab}, the parts which are tangent to a two-sphere $\sigma_a{}^c\sigma_b{}^d h_{cd}$ may be described by the trace with respect to σ_{ab} and by two potentials $h^{\rm ev}$ and $h^{\rm od}$ via

$$\begin{aligned}\sigma_a{}^c\sigma_b{}^d h_{cd} \;=\; & \frac{1}{2}h^{\rm trc}\sigma_{ab} + r^2 2\sigma_{(a}{}^c\sigma_{b)}{}^d\nabla_c\left(\sigma_d{}^e\nabla_e h^{\rm ev}\right) - r^2\sigma_{ab}\sigma^{cd}\nabla_c\left(\sigma_d{}^e\nabla_e h^{\rm ev}\right) \\ & - 2r^2\epsilon_{(a}{}^c\sigma_{b)}{}^d\nabla_c\left(\sigma_d{}^e\nabla_e h^{\rm od}\right)\end{aligned} \tag{23}$$

The potentials $h^{\rm ev}$ and $h^{\rm od}$ are generally functions of all of the spacetime coordinates and are guaranteed to exist by theorems involving solutions of elliptic equations on a two-sphere. The factors of r^2 are included for convenience.

The notation for a covariant tensor field is condensed by defining trace-free tensors tangent to a two-sphere and associated with the potential $h^{\rm ev}$

$$h^{\rm ev}_{ab} \equiv r^2 2\sigma_{(a}{}^c\sigma_{b)}{}^d\nabla_c\left(\sigma_d{}^e\nabla_e h^{\rm ev}\right) - r^2\sigma_{ab}\sigma^{cd}\nabla_c\left(\sigma_d{}^e\nabla_e h^{\rm ev}\right) \tag{24}$$

and with the potential $h^{\rm od}$

$$h^{\rm od}_{ab} \equiv -2r^2\epsilon_{(a}{}^c\sigma_{b)}{}^d\nabla_c\left(\sigma_d{}^e\nabla_e h^{\rm od}\right). \tag{25}$$

The ten independent components of a symmetric covariant tensor h_{ab} in a spherically symmetric geometry may be written as a sum of the form

$$\begin{aligned}h_{ab}\,{\rm d}x^a\,{\rm d}x^b \;=\; & h_{tt}\,{\rm d}t^2 + 2h_{tr}\,{\rm d}t\,{\rm d}r + 2\left(h^{\rm ev}_{tA} + h^{\rm od}_{tA}\right)\,{\rm d}t\,{\rm d}x^A \\ & + h_{rr}\,{\rm d}r^2 + 2\left(h^{\rm ev}_{rA} + h^{\rm od}_{rA}\right)\,{\rm d}r\,{\rm d}x^A \\ & + \left(h^{\rm trc}\sigma_{AB} + h^{\rm ev}_{AB} + h^{\rm od}_{AB}\right)\,{\rm d}x^A\,{\rm d}x^B\end{aligned} \tag{26}$$

in terms of the ten functions h_{tt}, h_{tr}, $h^{\rm ev}_t$, $h^{\rm od}_t$, h_{rr}, $h^{\rm ev}_r$, $h^{\rm od}_r$, $h^{\rm trc}$, $h^{\rm ev}$ and $h^{\rm od}$, As with the vector field, the capital indices A and B are used here just as a reminder that the vector or tensor to which they are attached is tangent to the two-sphere. Otherwise, they should be considered ordinary spacetime indices in the covariant spirit of Eq. (23)-Eq. (25).

The descriptions of vector and tensor potentials in Eqs. (19) and (23) on a two-sphere could have been written with a derivative operator involving the usual angular coordinates.

However, this would cloud the covariant nature of the decomposition which is clearly revealed above.

The description of the vector and tensor components in terms of potentials takes advantage of the the natural symmetry of the background geometry. For example, if a potential is a function of r and t times a $Y_{\ell m}$ then the resulting vector or tensor field is the same function times the vector or tensor spherical harmonic with the same ℓ, m pair. Expressions such as the perturbed Einstein tensor take a particularly simple form when written in terms of the potentials in place of the components.

III. INTEGRATION OF THE SCALAR WAVE EQUATION IN THE SCHWARZSCHILD GEOMETRY

The scalar field resulting from a charge q moving in a circular orbit of the Schwarzschild geometry provides an elementary example which contains many of the same interesting features of the metric perturbations. The wave equation for the scalar field is

$$\nabla^2\psi = -4\pi\rho, \tag{27}$$

where the scalar field source ρ represents a point charge q moving through spacetime along a worldline described by coordinates $z^a(\tau)$. This source is

$$\begin{aligned}\rho(x) &= q\int(-g)^{-1/2}\delta^4(x^a - z^a(\tau))d\tau \\ &= q(-g)^{-1/2}(dt/d\tau)^{-1}\delta^3(x^i - z^i(t)),\end{aligned} \tag{28}$$

with τ being the proper time along the worldline. For a circular orbit at radius R, expanding ρ in terms of spherical harmonic components provides

$$\begin{aligned}\rho &= q\int(-g)^{-1/2}\delta(r-R)\delta(\theta-\pi/2)\delta(\phi-\Omega t)\delta(t-t(\tau))d\tau \\ &= r^{-2}q\delta(r-R)\delta(\theta-\pi/2)\delta(\phi-\Omega t)dt/d\tau \\ &= \sum_{\ell m}\frac{q_{\ell m}}{4\pi R}\delta(r-R)e^{i\omega_m t}Y_{\ell m}(\theta,\phi),\end{aligned} \tag{29}$$

where

$$\omega_m \equiv -m\Omega, \tag{30}$$

$$q_{\ell m} = \frac{4\pi q}{R}\frac{Y^*_{\ell m}(\pi/2,0)}{dt/d\tau}, \tag{31}$$

and

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-3M/R}}, \tag{32}$$

which follows from the knowledge of the circular geodesics of the Schwarzschild geometry.

Also, decomposing ψ provides

$$\psi = \sum_{\ell,m}\frac{\psi_{\ell m}(r)}{r}e^{i\omega_m t}Y_{\ell m}(\theta,\phi), \tag{33}$$

and the ℓm component of the scalar wave equation becomes

$$\frac{d^2(\psi_{\ell m}/r)}{dr^2}+\frac{2(r-M)}{r(r-2M)}\frac{d(\psi_{\ell m}/r)}{dr}+\left[\frac{\omega^2 r^2}{(r-2M)^2}-\frac{\ell(\ell+1)}{r(r-2M)}\right](\psi_{\ell m}/r)=-\frac{q_{\ell m}}{R-2M}\delta(r-R). \tag{34}$$

This may also be written in terms of the "tortoise" coordinate

$$r_* = r + 2M\ln(r/2M-1) \quad \text{and} \quad dr_*/dr = r/(r-2M) \tag{35}$$

as

$$\frac{d^2\psi_{\ell m}}{dr_*^2}+\left[\omega^2-\left(1-\frac{2M}{r}\right)\frac{\ell(\ell+1)+2M/r}{r^2}\right]\psi_{\ell m} = -4\pi(1-3M/R)^{-1/2}qY_{\ell m}(\pi/2,0)\delta(r-R). \tag{36}$$

We know that

$$Y_{\ell,-m} = (-1)^m Y^*_{\ell,m}, \tag{37}$$

and the reality of ρ and of the final solution for $\psi(t,r,\theta,\phi)$ requires similar expressions for $q_{\ell,-m}$ and $\psi_{\ell,-m}$.

The boundary conditions of interest require only downgoing waves at the event horizon and we let one homogeneous solution of Eq. (34) be

$$\psi^H_{\ell m} = e^{i\omega r_*}, \quad r\to 2M. \tag{38}$$

A second homogeneous solution of Eq. (34) with only outgoing waves at infinity is

$$\psi^\infty_{\ell m} = e^{-i\omega r_*} \quad r\to\infty. \tag{39}$$

We can numerically integrate Eq. (34) from very near the event horizon out to the radius of the orbit R to give ψ^H, and we can similarly integrate Eq. (34) from "near" infinity in to R to give ψ^∞.

The retarded field is then given as a multiple of ψ^H for $r<R$ and as a multiple of ψ^∞ for $r>R$,

$$\psi^{\rm ret}_{\ell m} = \begin{cases} A\,\psi^H_{\ell m}, & r<R \\ B\,\psi^\infty_{\ell m}, & r>R. \end{cases} \tag{40}$$

The coefficients A and B are determined both by properly matching the discontinuity in $d\psi_{\ell m}/dr$ at R with the δ-function source and also by requiring that $\psi_{\ell m}$ be continuous at R. The discontinuity in $d\psi_{\ell m}/dr$ at R is

$$\left(B\frac{d\psi^\infty_{\ell m}}{dr}-A\frac{d\psi^H_{\ell m}}{dr}\right)_R = -\frac{q_{\ell m}}{R-2M}, \tag{41}$$

and the continuity then yields

$$A\left(\psi^H_{\ell m}\frac{d\psi^\infty_{\ell m}}{dr}-\psi^\infty_{\ell m}\frac{d\psi^H_{\ell m}}{dr}\right) = -\psi^\infty_{\ell m}\frac{q_{\ell m}}{R-2M}, \tag{42}$$

and

$$B\left(\psi^H_{\ell m}\frac{d\psi^\infty_{\ell m}}{dr}-\psi^\infty_{\ell m}\frac{d\psi^H_{\ell m}}{dr}\right) = -\psi^H_{\ell m}\frac{q_{\ell m}}{R-2M}. \tag{43}$$

IV. PERTURBATIONS OF THE SCHWARZSCHILD GEOMETRY

The "Regge-Wheeler equation" for odd parity, $\ell \geq 2$, source-free metric perturbations of the Schwarzschild geometry is

$$\frac{d^2\psi_{\ell m}}{dr_*^2} + \left[\omega^2 - \left(1 - \frac{2M}{r}\right)\frac{\ell(\ell+1) - 6M/r}{r^2}\right]\psi_{\ell m} = 0. \tag{44}$$

The even parity perturbations with $\ell \geq 2$ are described in a similar manner by the "Zerilli equation" which is algebraically more complex than Eq. (44), but structurally very similar. In fact, any homogeneous solution of the Zerilli equation may be written as a fairly simple linear combination of a solution of the Regge-Wheeler equation and its derivative. And the magnitudes of the reflection and transmission amplitudes of these rather different equations are, in fact identical.

A. Free oscillations of a black hole

The free oscillations of a black hole are variously described as the "quasi-normal modes" or the "ring-down" of the hole. These oscillations are mathematically described as a homogeneous solution of, say, the Regge-Wheeler equation which has only outgoing radiation in the limit that $r \to \infty$ and simultaneously only downgoing radiation in the limit that $r \to 2M$.

An equivalent description of a free oscillation is in terms of the scattering amplitude. Focus on a gravitational wave, of a specific frequency, being sent in from infinity toward the black hole with a unit amplitude. Some of the wave is absorbed by the black hole and some is reflected. Thus

$$\psi_{\ell m} = \begin{cases} e^{-i\omega r_*} + B_{\rm ref} e^{i\omega r_*}, & r_* \to +\infty \\ A_{\rm abs} e^{-i\omega r_*}, & r_* \to -\infty. \end{cases} \tag{45}$$

The complex amplitudes $A_{\rm abs}$ and $B_{\rm ref}$ are complex functions of the complex frequency. A frequency of a free oscillation is then characterized as being a pole in $A_{\rm abs}$ —that characteristic frequency has a solution to the wave equation with purely outgoing waves at infinity and downgoing waves at the event horizon. The imaginary part of the eigenfrequency gives the e-folding time of the damping, and the real part gives the frequency of the oscillations.

The frequencies of the even and the odd parity free oscillations of a black hole are identical, and were first calculated back in the 70's [5, 6].

<u>Exercise</u> 1: Show that the separated equation for $\psi(r)$, in the usual Schwarzschild coordinates, is the result given in Eq. (36)

<u>Exercise</u> 2: Consider the toy problem, which uses "Price's potential", where the separated equation for $\psi(r_*)$ is

$$\frac{d^2\psi_{\ell m}}{dr_*^2} + \left[\omega^2 - V(r_*)\right]\psi_{\ell m} = 0 \tag{46}$$

over the range $-\infty < r_* < +\infty$, and where

$$\begin{aligned} V(r_*) &= O \quad \text{for } r_* < 3M \\ &= \frac{\ell(\ell+1)}{r_*^2} \quad \text{for } r_* \geq 3M. \end{aligned} \tag{47}$$

Recall that "quasi-normal modes" are those solutions to the vacuum wave equation which have outgoing waves at large r_*, *i.e.* $\psi_{\ell m}(r_*) \sim \exp(-i\omega r_*)$ as $r_* \to +\infty$, and downgoing waves at the event horizon, *i.e.* $\psi_{\ell m}(r_*) \sim \exp(i\omega r_*)$ as $r_* \to -\infty$. Finding the quasi-normal modes is, thus, an eigenvalue problem for the complex frequencies ω whose wave functions Ψ satisfy appropriate boundary conditions. Find all of the eigen-frequencies of these "quasi-normal modes" for $\ell = 0$, 1 and 2.

Hint: Use $\psi_{\ell m}(r_*) \sim \exp(i\omega r_*)$ for $r_* < 3M$ and $\psi_{\ell m}(r_*) \sim$ a spherical Bessel function for $r_* > 3M$. Scale $\psi_{\ell m}$ so that it is continuous at $r_* = 3M$ and then find ω such that the $d\psi_{\ell m}/dr_*$ is also continuous at $r_* = 3M$.

<u>Exercise</u> 3: With Price's potential (rather than the actual potential of Eq. (36)), let a point source with a scalar charge q be in a circular orbit at radius $r = R$ with orbital frequency $\Omega = (M/R^3)^{1/2}$. Calculate the amplitude of the scalar radiation that goes out at infinity and that goes down the black hole for all values of $m^2 \leq \ell$ and $\ell = 0$ or 1. For simplicity, assume that the ℓm component of the wave equation takes the form

$$\frac{d^2\psi_{\ell m}}{dr_*^2} + \left[\omega^2 - V(r_*)\right]\psi_{\ell m} = q_{\ell m}\delta(r - R). \tag{48}$$

To make this problem a bit less complicated, you may assume that the radius of the orbit of the scalar charge is at $R = 3M$.

V. GAUGE ISSUES

A. Gauge transformations

In perturbation analyses of general relativity [7–9], one considers the difference in the actual metric $g^{\rm act}_{ab}$ of an interesting, perturbed spacetime and the abstract metric g^0_{ab} of some given, background spacetime. The difference

$$h_{ab} = g^{\rm act}_{ab} - g^0_{ab} \tag{49}$$

is assumed to be infinitesimal, say O(h). Typically, one determines a set of linear equations which govern h_{ab} by expanding the Einstein equations through O(h). The results are often used to resolve interesting issues concerning the stability of the background, or the propagation and emission of gravitational waves by a perturbing source.

However, Eq. (49) is ambiguous: The metrics $g^{\rm act}_{ab}$ and g^0_{ab} are given on different manifolds. For a given event on one manifold at which corresponding event on the other manifold is the subtraction to be performed? Usually a coordinate system common to both spacetimes induces an implicit mapping between the manifolds and defines the subtraction. Yet, the presence of the perturbation allows ambiguity. An infinitesimal coordinate transformation of the perturbed spacetime

$$x'^a = x^a + \xi^a, \quad \text{where} \quad \xi^a = \mathrm{O}(h), \tag{50}$$

not only changes the components of a tensor at O(h), in the usual way, but also changes the mapping between the two manifolds in Eq. (49). After the transformation Eq. (50),

$$h^{\rm new}_{ab} \;=\; \left(g^0_{cd} + h^{\rm old}_{cd}\right)\frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} - \left(g^0_{ab} + \xi^c\frac{\partial g^0_{ab}}{\partial x^c}\right). \tag{51}$$

The ξ^c in the last term accounts for the O(h) change in the event of the background used in the subtraction. After an expansion, this provides a new description of h_{ab}

$$\begin{aligned} h^{\rm new}_{ab} &= h^{\rm old}_{ab} - g^0_{cb}\frac{\partial \xi^c}{\partial x^a} - g^0_{cb}\frac{\partial \xi^d}{\partial x^b} - \xi^c \frac{\partial g^0_{ab}}{\partial x^c} \\ &= h^{\rm old}_{ab} - \mathcal{L}_\xi g^0_{ab} = h^{\rm old}_{ab} - 2\nabla_{(a}\xi_{b)} \end{aligned} \tag{52}$$

through O(h); the symbol $\mathcal{L}$ represents the Lie derivative and ∇_a is the covariant derivative compatible with g^0_{ab}.

The action of such an infinitesimal coordinate transformation is called a *gauge transformation* and does not change the actual perturbed manifold, but it does change the coordinate description of the perturbed manifold.

A similar circumstance holds with general coordinate transformations. A change in coordinate system creates a change in description. But, general covariance dictates that actual physical measurements must be describable in a manner which is invariant under a change in coordinates. Thus, one usually describes physically interesting quantities strictly in terms of geometrical scalars which, by nature, are coordinate independent.

In a perturbation analysis any physically interesting result ought to be describable in a manner which is gauge invariant.

B. Gauge invariant quantities

Gauge invariant quantities appear to fall into a few different categories.

The change in any geometrical quantity under a gauge transformation is determined by the Lie derivative of that same quantity on the background manifold. This is demonstrated for the gauge transformation of a metric perturbation in Eq. (52). Thus, if a geometrical quantity vanishes in the background, but not in the perturbed metric, then it will be gauge invariant. Examples include the Newman-Penrose scalars Ψ_0 and Ψ_4 which vanish for the Kerr metric. In the perturbed Kerr metric Ψ_0 and Ψ_4 are non zero, gauge invariant and the basis for perturbation analyses of rotating black holes. A second example has the background metric being a vacuum solution of the Einstein equations, so its Ricci tensor R_{ab} vanishes. The Ricci tensor of a perturbation of this metric is then unchanged by a gauge transformation. This is directly demonstrated in IX.

Some quantities which are associated with a symmetry of the perturbed geometry are gauge invariant. For example a geodesic of a perturbed Schwarzschild metric, where the perturbation is axisymmetric with Killing field k^a, has a constant of motion $k^a u^b (g^0_{ab} + h_{ab})$ which is gauge invariant.

Another symmetry example involves the Schwarzschild geometry with an arbitrary perturbation. It is a fact that a gauge transformation can always be found, such that the resulting h_{ab} has the components $h_{\theta\theta}$, $h_{\theta\phi}$ and $h_{\phi\phi}$ all equal to zero. In this gauge, the surfaces of constant r and t are geometrical two-spheres, even while the manifold as whole has no symmetry. The area of each two-sphere can be used to define a radial scalar field R which is constant on each of these two-spheres. This scalar field on the perturbed Schwarzschild manifold is independent of gauge. However, its coordinate description in terms of the usual t, r, θ, ϕ coordinates does change under a gauge transformation.

Quantities which are carefully described by a physical measurement are gauge invariant. For example, the acceleration of a world line could be measured with masses and springs by an observer moving along a world line in a perturbed geometry. The magnitude of the acceleration is a scalar and is gauge invariant. If the world line has zero acceleration, then it is a geodesic. Therefore, a geodesic of a perturbed metric remains a geodesic under a gauge transformation even while its coordinate description changes by $\mathrm{O}(h)$.

The mass and angular momentum are other gauge invariant quantities which might be measured by distant observers in an asymptotically flat spacetime. A small mass orbiting a larger black hole perturbs the black hole metric and emits gravitational waves. The gravitational waveform measured at a large distance is also gauge invariant.

VI. PERTURBED BIANCHI IDENTITY

The Bianchi identity is

$$\nabla_c R_{dea}{}^b + \nabla_e R_{cda}{}^b + \nabla_d R_{eca}{}^b = 0. \tag{53}$$

Contraction on c and b implies that

$$\nabla_b R_{dea}{}^b = 0 \tag{54}$$

for a vacuum solution of the Einstein equations. This result is used often in the derivations of identities involving $E_{ab}(h)$.

The definition of the operator E_{ab} for a vacuum spacetime is

$$2E_{ab}(h) = \nabla^2 h_{ab} + \nabla_a\nabla_b h - 2\nabla_{(a}\nabla^c h_{b)c} + 2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c\nabla^d h_{cd} - \nabla^2 h), \tag{55}$$

so that

$$\begin{aligned} 2\nabla^a E_{ab}(h) &= \nabla^a\nabla^c\nabla_c h_{ab} + \nabla^a\nabla_a\nabla_b h - \nabla^a\nabla_a\nabla^c h_{bc} - \nabla^a\nabla_b\nabla^c h_{ac} \\ &\quad + 2(\nabla^a R_a{}^c{}_b{}^d)h_{cd} + 2R^{ac}{}_b{}^d\nabla_a h_{cd} + \nabla_b\nabla^c\nabla^d h_{cd} - \nabla_b\nabla^c\nabla_c h. \\ &= \nabla^a\nabla^c\nabla_a h_{cb} - \nabla^a\nabla_a\nabla^c h_{bc} - \nabla^a\nabla_b\nabla^c h_{ac} + \nabla_b\nabla^a\nabla^c h_{ac} \\ &\quad + R^{ac}{}_b{}^d\nabla_c h_{ad} + 2R^{ac}{}_b{}^d\nabla_a h_{cd} \\ &= 0. \end{aligned} \tag{56}$$

The second equality follows after use of the Ricci identity to interchange the order of derivatives on the first, second and last terms as well as repeated uses of $R_{ab} = 0$ and Eq. (54) for vacuum spacetimes. The final result follows after use of the Ricci identity on the first two terms and on the second two terms of the second equality, and the application of symmetries of the Riemann tensor on the remainder.

If h_{ab} is not C^3 then $\nabla^a E_{ab}(h) = 0$ in a distributional sense. To show this, choose an arbitrary, smooth test vector field λ^a with compact support. Consider the integral of $\lambda^b\nabla^a E_{ab}(h)$ over a sufficiently large region. Integrate by parts once and discard the surface term. Next use Eq. (77) and discard the surface terms to obtain an integral of $h^{ab}E_{ab}(\nabla\lambda)$. This integral is zero from Eq. (68). These steps also provide an alternative derivation of Eq. (56) in the event that h_{ab} is C^3, as well.

VII. FORMAL nTH ORDER PERTURBATION ANALYSIS

In general perturbation analysis, let the g_{ab} of Eq. (3) be an exact solution to the vacuum Einstein equations, g^0_{ab}, and iteratively define

$$g^{(n)}_{ab} = g^{(n-1)}_{ab} + h^{(n)}_{ab} \tag{57}$$

where

$$h^{(n)}_{ab} = \mathrm{O}(\mu^n). \tag{58}$$

Assume that we are given $g^{(n-1)}_{ab}$ and $T^{(n)}_{ab} = \mathrm{O}(\mu)$, with

$$G^{(n-1)}_{ab} - 8\pi T^{(n)}_{ab} = \mathrm{O}(\mu^n). \tag{59}$$

If $h^{(n)}_{ab}$ is a solution of Eq. (59) from

$$E_{ab}(h^{(n)}) = G^{(n-1)}_{ab} - 8\pi T^{(n)}_{ab} + \mathrm{O}(\mu^{n+1}), \tag{60}$$

then it follows from the definition of the operator $E_{ab}(h)$ in Eq. (2) that

$$G^{(n)}_{ab} - 8\pi T^{(n)}_{ab} = \mathrm{O}(\mu^{n+1}), \tag{61}$$

and $h^{(n)}_{ab}$ is an $\mathrm{O}(\mu^n)$ improvement to the approximate solution to the Einstein equations.

The Bianchi identity implies that

$$\nabla^a E_{ab}(h) = 0 \tag{62}$$

for any symmetric C^3 tensor field h_{ab}, as shown in VI. It is also shown that if h_{ab} is not C^3 then Eq. (62) holds in a distributional sense. Thus an integrability condition of Eq. (60) is that

$$\nabla^a \big(G^{(n-1)}_{ab} - 8\pi T^{(n)}_{ab}\big) = \mathrm{O}(\mu^{n+1}). \tag{63}$$

Note, however, that

$$\begin{aligned}\nabla^a \big(G^{(n-1)}_{ab} - 8\pi T^{(n)}_{ab}\big) &= \nabla^a_{(n-1)} \big(G^{(n-1)}_{ab} - 8\pi T^{(n)}_{ab}\big) \\ &\quad + \Gamma^a_{ac}\big(G^{(n-1)c}_{b} - 8\pi T^{(n)c}_{b}\big) - \Gamma^c_{ab}\big(G^{(n-1)a}_{c} - 8\pi T^{(n)a}_{c}\big),\end{aligned} \tag{64}$$

where $\nabla^a_{(n-1)}$ is the derivative operator of $g^{(n-1)}_{ab}$, and Γ^a_{bc} is the connection relating the derivative operators ∇^a and $\nabla^a_{(n-1)}$. The Bianchi identity implies that

$$\nabla^a_{(n-1)} G^{(n-1)}_{ab} = 0, \tag{65}$$

and the terms in Eq. (64) involving Γ^a_{bc} are order μ^{n+1} because of Eq. (59) and the fact that $\Gamma^a_{bc} = \mathrm{O}(\mu)$. Thus, the approximate vanishing of the right hand side of Eq. (64) is the integrability condition for Eq. (60),

$$\nabla^a_{(n-1)} T^{(n)}_{ab} = \mathrm{O}(\mu^{n+1}). \tag{66}$$

In other words, before Eq. (60) can be solved for $h^{(n)}_{ab}$, it is necessary that the perturbing stress tensor be adjusted to be conserved with the metric $g^{(n-1)}_{ab}$ and to satisfy Eq. (66).

VIII. $\nabla_b T^{ab} = 0$ IMPLIES THE GEODESIC EQUATION FOR A POINT MASS.

We follow an example in reference [10]. In Eq. (8), $\delta^4(x^a - X^a(s))/\sqrt{-g}$ is a scalar field, and the factor u^b may be defined as a vector field by extension, in any smooth manner, away from the world line. Then,

$$\begin{aligned}(g^c{}_a + u^c u_a)\nabla_b T^{ab} &= \mu(g^c{}_a + u^c u_a)\int_{-\infty}^{\infty}\left[\frac{(\nabla_b u^a)u^b}{\sqrt{-g}}\delta^4(x^a - X^a(s))\right.\\ &\quad \left. + u^a\nabla_b\left(\frac{u^b}{\sqrt{-g}}\delta^4(x^a - X^a(s))\right)\right]\mathrm{d}s\\ &= \mu\int_{-\infty}^{\infty}\frac{(\nabla_b u^a)u^b}{\sqrt{-g}}\delta^4(x^a - X^a(s))\,\mathrm{d}s \qquad (67)\end{aligned}$$

where the second equality follows from properties of the projection operator $g^c{}_a + u^c u_a$. If $\nabla_b T^{ab} = 0$, then it necessarily follows that the coefficient of the delta function must be zero for all proper times. A consequence is that $u^b \nabla_b u^a = 0$, the geodesic equation.

A more formal proof of this result is in Poisson's review of the self-force [11], p 89.

IX. GAUGE INVARIANCE OF $E_{ab}(h)$

For a background geometry which is a vacuum solution of the Einstein equations, an infinitesimal gauge transformation, $x^a_{\mathrm{new}} = x^a + \xi^a$, with $\xi^a = \mathrm{O}(\mu)$ changes the metric perturbation, $h^{\mathrm{new}}_{ab} = h_{ab} - 2\nabla_{(a}\xi_{b)} + \mathrm{O}(\mu^2)$. But the operator $E_{ab}(h)$ is invariant under such a coordinate transformation,

$$E_{ab}(\nabla\xi) = 0. \qquad (68)$$

This result follows immediately from the fact that the change in the perturbation of the Einstein tensor E_{ab} under a gauge transformation is the Lie derivative of the background Einstein tensor $\mathcal{L}_\xi G_{ab}$. For a vacuum background spacetime, this is zero.

Equation (68) also follows from direct substitution into

$$2E_{ab}(h) = \nabla^2 h_{ab} + \nabla_a\nabla_b h - 2\nabla_{(a}\nabla^c h_{b)c} + 2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c\nabla^d h_{cd} - \nabla^2 h) \qquad (69)$$

with $h_{ab} = 2\nabla_{(a}\xi_{b)}$. It is easiest to consider the factor of g_{ab} separately,

$$\begin{aligned}\text{factor of } g_{ab} &= \nabla^c\nabla^d\nabla_c\xi_d + \nabla^c\nabla^d\nabla_d\xi_c - 2\nabla^a\nabla_a\nabla^b\xi_b\\ &= 2\nabla^c\nabla^d\nabla_c\xi_d - 2\nabla^a\nabla_a\nabla^b\xi_b\\ &= 0. \qquad (70)\end{aligned}$$

The second equality follows after use of the Ricci identity on the first two indices of the second term, use of $R_{ab} = 0$ for a vacuum spacetime and a relabeling of the indices. The final result follows after use of the Ricci identity on the second term of the second equality and use of $R_{ab} = 0$ for a vacuum spacetime. With $h_{ab} = 2\nabla_{(a}\xi_{b)}$, the remainder of $E_{ab}(2\nabla\xi)$ is

$$\begin{aligned}\text{remainder} &= \nabla^c\nabla_c\nabla_a\xi_b + \nabla^c\nabla_c\nabla_b\xi_a + 2\nabla_a\nabla_b\nabla^c\xi_c\\ &\quad - \nabla_a\nabla^c\nabla_b\xi_c - \nabla_a\nabla^c\nabla_c\xi_b - \nabla_b\nabla^c\nabla_a\xi_c - \nabla_b\nabla^c\nabla_c\xi_a\\ &\quad + 2R_a{}^c{}_b{}^d\nabla_c\xi_d + 2R_a{}^c{}_b{}^d\nabla_d\xi_c. \qquad (71)\end{aligned}$$

The analysis of this expression is lengthy but not difficult. It begins with using the Ricci identity upon the second and third indices of the first, second, fourth and sixth terms and upon the first and second indices of the fifth and seventh terms. The resulting terms with three derivatives may be paired up in a way to use the Ricci identity again and to reduce the entire expression to one involving only single derivatives. This also requires application of Eq. (54). That the entire expression is zero, then follows from the symmetries of the Riemann tensor.

X. GREEN'S THEOREM FOR E_{ab}

Assume that the background geometry is a vacuum spacetime; *i.e.* $R_{ab} = 0 = R$. The operator $E_{ab}(h)$ in Eq. (3), with an arbitrary tensor k^{ab}, satisfies the identity

$$2k^{ab}E_{ab}(h) = \nabla_c F^c(k,h) - \left\langle k^{ab}, h_{ab}\right\rangle, \tag{72}$$

where

$$F^c(k,h) \;\equiv\; k^{ab}\nabla^c h_{ab} - \frac{1}{2}k\nabla^c h - 2(k^{cb} - \frac{1}{2}g^{cb}k)\nabla^a(h_{ab} - \frac{1}{2}g_{ab}h) \tag{73}$$

or, in terms of $\bar{h}_{ab} \equiv h_{ab} - \frac{1}{2}g_{ab}h^c{}_c$,

$$F^c(k,h) \;\equiv\; \bar{k}^{ab}\nabla^c\bar{h}_{ab} - 2\bar{k}^{cb}\nabla^a\bar{h}_{ab} \tag{74}$$

and

$$\begin{aligned}\left\langle k^{ab}, h_{ab}\right\rangle \;\equiv\; & \nabla^c k^{ab}\nabla_c h_{ab} - \frac{1}{2}\nabla^c k\nabla_c h \\ & - 2\nabla_a\Big(k^{ac} - \frac{1}{2}g^{ac}k\Big)\nabla^b\Big(h_{bc} - \frac{1}{2}g_{bc}h\Big) - 2k^{ab}R_a{}^c{}_b{}^d h_{cd}.\end{aligned} \tag{75}$$

or

$$\begin{aligned}\left\langle k^{ab}, h_{ab}\right\rangle \;\equiv\; & \nabla^c\bar{k}^{ab}\nabla_c\bar{h}_{ab} \\ & - 2\nabla_a\bar{k}^{ac}\nabla^b\bar{h}_{bc} - 2\bar{k}^{ab}R_a{}^c{}_b{}^d\bar{h}_{cd}.\end{aligned} \tag{76}$$

Note that the "inner product," $\left\langle k^{ab}, h_{ab}\right\rangle = \left\langle h^{ab}, k_{ab}\right\rangle$ is symmetric under the interchange of h^{ab} and k_{ab}. It follows that

$$k^{ab}E_{ab}(h) - h^{ab}E_{ab}(k) = \frac{1}{2}\nabla_c\left[F^c(k,h) - F^c(h,k)\right]. \tag{77}$$

Which is a tensor version of Green's theorem for the differential operator $E_{ab}(h)$.

EXERCISE 4: **Assume that the background geometry is a vacuum solution of the Einstein equations.** From Eqs. (3) and (72), derive Eqs. (73) and (76). Hint: Contract Eq. (3) with an arbitrary symmetric tensor k^{ab}, and move k^{ab} inside ∇_a in each term by "differentiating by parts." The "boundary terms" constitute $F^c(k,h)$.

XI. GAUSS' LAW

Consider a compact region of an n-dimensional manifold, which has a boundary defined by some scalar field $t =$*const*. Let the unit normal to the boundary be

$$n_a = \epsilon N \nabla_a t = \epsilon \nabla_a t/(\epsilon g^{bc}\nabla_b t \nabla_c t)^{1/2}, \tag{78}$$

where ϵ is ± 1 depending upon whether n_a is spacelike or timelike, respectively.

It is useful to note a variety of forms of Gauss' law:

$$\begin{aligned} \int (\nabla_b A^b)\, \sqrt{g}\, \mathrm{d}^n x &= \int \frac{\partial}{\partial x^b}(\sqrt{g} A^b)\, \mathrm{d}^n x = \oint (\nabla_b t)\, A^b \sqrt{g}\, \mathrm{d}^{n-1} x \\ &= \oint \frac{n_b A^b}{N} \sqrt{g}\, \mathrm{d}^{n-1} x = \oint n_b A^b \sqrt{\gamma}\, \mathrm{d}^{n-1} x \end{aligned} \tag{79}$$

where the metric on the boundary is $\gamma_{ab} = g_{ab} - \epsilon n_a n_b$, and we use

$$|g_{\rm det}| = |\gamma_{\rm det} N^2|. \tag{80}$$

XII. SINGULAR GAUGE TRANSFORMATIONS

Let ξ^a be a, possibly distribution valued, vector field. And let $h_{ab} = -2\nabla_{(a}\xi_{b)}$, as for a gauge transformation. Also, let k_{ab} be a smooth "test" tensor with compact support. Then

$$\begin{aligned} \int k^{ab} E_{ab}(h) \sqrt{-g}\, \mathrm{d}^4 x &= \int h^{ab} E_{ab}(k) \sqrt{-g}\, \mathrm{d}^4 x \\ &= -2 \int (\nabla^a \xi^b) E_{ab}(k) \sqrt{-g}\, \mathrm{d}^4 x, \end{aligned} \tag{81}$$

from Eq. (77), after dropping the divergence term. An integration by parts and application of the perturbed Bianchi identity (56) yields

$$\begin{aligned} \int k^{ab} E_{ab}(h) \sqrt{-g}\, \mathrm{d}^4 x &= 2 \int \xi^b \nabla^a \left[E_{ab}(k) \right] \sqrt{-g}\, \mathrm{d}^4 x \\ &= 0. \end{aligned} \tag{82}$$

Thus, we demonstrate that given a solution to the inhomogeneous, perturbed Einstein equations Eq. (1), even a distributional gauge transformation leaves a distributional valued metric perturbation that continues to satisfy the perturbed Einstein equations.

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[13] Eq. (35.64) of MTW [12] is actually based upon $\delta R_{ab} = 0$, however for a perturbation of a non vacuum spacetime, it seems preferable to base the analysis on $\delta(2R_{ab} - g_{ab}R) = 16\pi\delta T_{ab}$. Note that if $\delta(2T_{ab} - g_{ab}T) = 0$, then $2\delta T_{ab} = -h_{ab}T + g_{ab}h^{cd}T_{cd}$. This explains the last two terms in Eq. (4) which are not in Eq. (35.64) of MTW