

Notes on the Lagrangian formulation of General Relativity

These notes are not a substitute in any manner for class lectures. These notes are now, essentially, complete. Please let me know if you find errors. Thanks, Steve Detweiler

I. LAGRANGIAN FORMULATION OF A CLASSICAL FIELD THEORY IN CURVED SPACETIME

The Lagrangian density \mathcal{L} of a classical, massless scalar field is given by

$$\mathcal{L}(\psi, \nabla_a \psi, g^{ab}) \equiv L\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g^{ab}\nabla_a\psi\nabla_b\psi, \quad (1)$$

while the action is

$$S = \int_{4\text{-vol}} L(\psi, \nabla_a \psi, g^{ab})\sqrt{-g} d^4x. \quad (2)$$

The assumption that the action is an extremum under arbitrary variations of ψ leads to the Euler-Lagrange equation,

$$\begin{aligned} \delta S_\psi &= \int_{4\text{-vol}} \left(\delta\psi \frac{\delta L}{\delta\psi} + \delta(\nabla_a\psi) \frac{\delta L}{\delta(\nabla_a\psi)} \right) \sqrt{-g} d^4x \\ &= \int_{4\text{-vol}} \nabla_a \left(\delta\psi \frac{\delta L}{\delta\nabla_a\psi} \right) \sqrt{-g} d^4x + \int_{4\text{-vol}} \delta\psi \left[\frac{\delta L}{\delta\psi} - \nabla_a \left(\frac{\delta L}{\delta\nabla_a\psi} \right) \right] \sqrt{-g} d^4x, \\ &= \oint_{3\text{-surf}} n_a \left(\delta\psi \frac{\delta L}{\delta\nabla_a\psi} \right) \sqrt{|h|} d^3x + \int_{4\text{-vol}} \delta\psi \left[\frac{\delta L}{\delta\psi} - \nabla_a \left(\frac{\delta L}{\delta\nabla_a\psi} \right) \right] \sqrt{-g} d^4x, \end{aligned} \quad (3)$$

where for the second equality we have commuted ∇_a and δ and integrated by parts, and the third equality follows by writing the volume integral of a divergence as a surface integral. n_a is the unit normal of the boundary of the 4-volume, and h_{ab} is the metric of the boundary. And the Euler-Lagrange equation

$$\begin{aligned} 0 &= \frac{\delta L}{\delta\psi} - \nabla_a \left(\frac{\delta L}{\delta\nabla_a\psi} \right) \\ &= \nabla_a (g^{ab}\nabla_b\psi) = 0 \\ &= \nabla^a\nabla_a\psi = 0 \end{aligned} \quad (4)$$

is the field equation for the massless scalar field. Note, however, that Eq. (1) is the appropriate Lagrangian density only if the boundary conditions specify either the value of ψ on the boundary or that the normal derivative of ψ is zero on the boundary; otherwise, the boundary integrals would not vanish.

We also expect that the action should be invariant under a change in the metric that results only from a change in the coordinate system. Thus we evaluate

$$\begin{aligned} \delta S_g &= \int_{4\text{-vol}} \left(\delta g^{ab} \frac{\delta L}{\delta g^{ab}} + \frac{1}{2} \frac{\delta g}{g} L \right) \sqrt{-g} d^4x \\ &= \int_{4\text{-vol}} \delta g^{ab} \left(\frac{\delta L}{\delta g^{ab}} - \frac{1}{2} g_{ab} L \right) \sqrt{-g} d^4x, \end{aligned} \quad (5)$$

where we have used $\delta g/g = g^{ab}\delta g_{ab} = -g_{ab}\delta g^{ab}$, which follows from elementary analysis of the determinant of a matrix. Now, δS_g is to vanish for arbitrary δg^{ab} which results from a coordinate transformation. In particular, for $x^a \rightarrow x^a + \xi^a$, where ξ^a is an infinitesimal change in the coordinates, $\delta g^{ab} = -\mathcal{L}_\xi g^{ab} = -2\nabla^{(a}\xi^{b)}$. Thus, for such a change in coordinates

$$\begin{aligned}\delta S_g &= \int_{4\text{-vol}} -2\nabla^a \xi^b \left(\frac{\delta L}{\delta g^{ab}} - \frac{1}{2}g_{ab}L \right) \sqrt{-g} d^4x \\ &= \int_{4\text{-vol}} 2\xi^b \nabla^a \left(\frac{\delta L}{\delta g^{ab}} - \frac{1}{2}g_{ab}L \right) \sqrt{-g} d^4x\end{aligned}\quad (6)$$

where we have integrated by parts, and dropped the surface term with the assumption that $\xi^b = 0$ on the boundary of the 4-volume. Thus the invariance of the action under an infinitesimal change in the coordinate system requires that

$$-\frac{1}{2}\sqrt{-g}T_{ab} \equiv \frac{\delta \mathcal{L}}{\delta g^{ab}} = \sqrt{-g} \left(\frac{\delta L}{\delta g^{ab}} - \frac{1}{2}g_{ab}L \right) \quad (7)$$

T_{ab} be divergence-free. T_{ab} is called the ‘‘stress-energy tensor’’ of the classical scalar field,

$$T_{ab} = \nabla_a \psi \nabla_b \psi - \frac{1}{2}g_{ab}(\nabla^c \psi) \nabla_c \psi. \quad (8)$$

II. LAGRANGIAN FORMULATION OF GENERAL RELATIVITY

The Lagrangian density of the gravitational field should be derived from a scalar which describes the geometry of spacetime, and we let

$$\mathcal{L}_g = \sqrt{-g}L_g = \frac{1}{16\pi}(R - 2\Lambda)\sqrt{-g} \quad (9)$$

where R is the scalar curvature of spacetime, and Λ is a constant which is usually called the ‘‘cosmological constant.’’ The corresponding action is

$$16\pi S = \int_{4\text{-vol}} (R - 2\Lambda)\sqrt{-g} d^4x. \quad (10)$$

This is not the only possible choice for a Lagrangian density, other scalars could be constructed from combinations of products of the Riemann and Ricci tensors. But, this choice leads to general relativity. With $R = g^{ab}R_{acb}{}^c$,

$$16\pi\delta L_g = \delta g^{ab}R_{acb}{}^c + g^{ab}\delta R_{acb}{}^c. \quad (11)$$

In Eq. (27) at the start of §V we show that this $\delta R_{acb}{}^c$ term reduces to a boundary integral, and the first term on the right hand side is $\delta g^{ab}R_{acb}{}^c = \delta g^{ab}R_{ab}$. It easily follows that the stress-energy tensor of the gravitational field is

$$-8\pi\sqrt{-g}T_{ab}^{\text{grav}} \equiv \frac{\delta \mathcal{L}_g}{\delta g^{ab}} = \sqrt{-g} \left(R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} \right) \quad (12)$$

And the action is an extremum under arbitrary variations of the the metric if and only if the Einstein equations with a cosmological constant

$$R_{ab} - \frac{1}{2}g_{ab}R + \Lambda g_{ab} = 8\pi T_{ab} \quad (13)$$

are satisfied.

EXERCISE 1: Fill in this last step to show that the Einstein equations, Eq. (13), really do follow from an action principle. Assume that the $\delta R_{acb}{}^c$ term, alluded to above, really does reduce to a boundary integral and can be ignored with an appropriate choice of boundary conditions.

III. DIFFERENTIAL GEOMETRY WITH TWO METRICS

We begin with an n dimensional manifold with two different metrics, g_{ab} and \bar{g}_{ab} , along with respective metric-compatible derivative operators, $\nabla_a g_{bc} = 0$ and $\bar{\nabla}_a \bar{g}_{bc} = 0$. First we see that $\nabla_a - \bar{\nabla}_a$ defines a connection tensor $\gamma^c{}_{ab}$. Given two arbitrary covariant vector fields, ξ_a and λ_a along with a scalar field α , consider

$$\begin{aligned} (\nabla_a - \bar{\nabla}_a)(\alpha\xi_b + \lambda_b) &= \nabla_a(\alpha\xi_b + \lambda_b) - \bar{\nabla}_a(\alpha\xi_b + \lambda_b) \\ &= (\nabla_a\alpha)\xi_b + \alpha\nabla_a\xi_b + \nabla_a\lambda_b - (\bar{\nabla}_a\alpha)\xi_b - \alpha\bar{\nabla}_a\xi_b - \bar{\nabla}_a\lambda_b \\ &= \alpha\nabla_a\xi_b + \nabla_a\lambda_b - \alpha\bar{\nabla}_a\xi_b - \bar{\nabla}_a\lambda_b \\ &= \alpha(\nabla_a - \bar{\nabla}_a)\xi_b + (\nabla_a - \bar{\nabla}_a)\lambda_b. \end{aligned} \quad (14)$$

Thus, we see that $\nabla_a - \bar{\nabla}_a$ corresponds to a linear operator from covariant vectors onto two-indexed covariant tensors. As we discussed last semester a linear operator from tensor fields to tensor fields is, in fact, a tensor field. And, we define the tensor $\gamma^c{}_{ab}$ by

$$\bar{\nabla}_a\xi_b = \nabla_a\xi_b - \gamma^c{}_{ab}\xi_c \quad (15)$$

for an arbitrary ξ_b . The rule for the difference of $\bar{\nabla}_a$ and ∇_a when operating on tensors of other types follows the same rule as the covariant derivative, partial derivative and the Christoffel symbol.

A constructive expression for $\gamma^c{}_{ab}$ is easily found by following a procedure that is very similar to that which we used to find the Christoffel symbols in the first set of notes from last semester. From the definition of a metric-compatible derivative operator

$$\bar{\nabla}_a\bar{g}_{bc} = 0 = \nabla_a\bar{g}_{bc} - \gamma^d{}_{ab}\bar{g}_{dc} - \gamma^d{}_{ac}\bar{g}_{bd}, \quad (16)$$

so that

$$\begin{aligned} \nabla_a\bar{g}_{bc} &= \gamma^d{}_{ab}\bar{g}_{dc} + \gamma^d{}_{ac}\bar{g}_{bd}, \\ \nabla_b\bar{g}_{ca} &= \gamma^d{}_{bc}\bar{g}_{da} + \gamma^d{}_{ba}\bar{g}_{cd}, \\ \nabla_c\bar{g}_{ab} &= \gamma^d{}_{ca}\bar{g}_{db} + \gamma^d{}_{cb}\bar{g}_{ad}. \end{aligned} \quad (17)$$

The second two of these follow from the first with cyclic permutations of the indices. Add the first two and subtract the third to obtain

$$\nabla_a\bar{g}_{bc} + \nabla_b\bar{g}_{ca} - \nabla_c\bar{g}_{ab} = 2\gamma^d{}_{ab}\bar{g}_{dc}, \quad (18)$$

where the symmetry of the Christoffel symbols is used. Now, raising the c index results in

$$\gamma^d{}_{ab} = \frac{1}{2}\bar{g}^{dc} (\nabla_a \bar{g}_{bc} + \nabla_b \bar{g}_{ca} - \nabla_c \bar{g}_{ab}). \quad (19)$$

The relationship between the Riemann tensors for the two metrics also follows the derivation of the Riemann tensor in terms of the derivatives of the Christoffel symbols in the first set of notes. Let ξ_a be an arbitrary vector field, and consider

$$\begin{aligned} \bar{\nabla}_a \bar{\nabla}_b \xi_c - \bar{\nabla}_b \bar{\nabla}_a \xi_c &= \nabla_a (\nabla_b \xi_c - \gamma^d{}_{bc} \xi_d) - \gamma^e{}_{ab} (\nabla_e \xi_c - \gamma^d{}_{ec} \xi_d) - \gamma^e{}_{ac} (\nabla_b \xi_e - \gamma^d{}_{be} \xi_d) \\ &\quad - \nabla_b (\nabla_a \xi_c - \gamma^d{}_{ac} \xi_d) + \gamma^e{}_{ba} (\nabla_e \xi_c - \gamma^d{}_{ec} \xi_d) + \gamma^e{}_{bc} (\nabla_a \xi_e - \gamma^d{}_{ae} \xi_d) \\ &= R_{abc}{}^d \xi_d - \nabla_a (\gamma^d{}_{bc} \xi_d) - \gamma^e{}_{ab} (\nabla_e \xi_c - \gamma^d{}_{ec} \xi_d) - \gamma^e{}_{ac} (\nabla_b \xi_e - \gamma^d{}_{be} \xi_d) \\ &\quad + \nabla_b (\gamma^d{}_{ac} \xi_d) + \gamma^e{}_{ba} (\nabla_e \xi_c - \gamma^d{}_{ec} \xi_d) + \gamma^e{}_{bc} (\nabla_a \xi_e - \gamma^d{}_{ae} \xi_d) \\ &= R_{abc}{}^d \xi_d - \nabla_a (\gamma^d{}_{bc} \xi_d) - \gamma^e{}_{ac} (\nabla_b \xi_e - \gamma^d{}_{be} \xi_d) \\ &\quad + \nabla_b (\gamma^d{}_{ac} \xi_d) + \gamma^e{}_{bc} (\nabla_a \xi_e - \gamma^d{}_{ae} \xi_d) \\ &= R_{abc}{}^d \xi_d - \nabla_a (\gamma^d{}_{bc}) \xi_d + \gamma^e{}_{ac} \gamma^d{}_{be} \xi_d \\ &\quad + \nabla_b (\gamma^d{}_{ac}) \xi_d - \gamma^e{}_{bc} \gamma^d{}_{ae} \xi_d \\ &= R_{abc}{}^d \xi_d - \nabla_a (\gamma^d{}_{bc}) \xi_d + \nabla_b (\gamma^d{}_{ac}) \xi_d - \gamma^e{}_{bc} \gamma^d{}_{ae} \xi_d + \gamma^e{}_{ac} \gamma^d{}_{be} \xi_d, \end{aligned} \quad (20)$$

where the first equality follows from the description of the difference of the covariant derivatives in terms of $\gamma^a{}_{bc}$, the second from the Ricci identity for the two different derivative operators, the third from the symmetry of $\gamma^a{}_{bc}$, the fourth from the Leibnitz rule for differentiation, and the fifth by rearranging terms. From the Ricci identity we also have

$$\bar{\nabla}_a \bar{\nabla}_b \xi_c - \bar{\nabla}_b \bar{\nabla}_a \xi_c = \bar{R}_{abc}{}^d \xi_d. \quad (21)$$

With ξ_a being arbitrary, it is necessary that

$$\bar{R}_{abc}{}^d = R_{abc}{}^d - \nabla_a \gamma^d{}_{bc} + \nabla_b \gamma^d{}_{ac} - \gamma^d{}_{ea} \gamma^e{}_{bc} + \gamma^d{}_{eb} \gamma^e{}_{ac}, \quad (22)$$

after a rearrangement of terms and indices.

Eq. (19) and (22) have many different applications such as with conformal transformations or, of immediate use, for perturbation analysis.

IV. PERTURBATION ANALYSIS

If the two metrics from the preceding section are only infinitesimally different, then $\delta g_{ab} \equiv g_{ab} - \bar{g}_{ab}$ is assumed to be small, and we work only through first order in δg_{ab} throughout this section. From Eq. (19)

$$\gamma^d{}_{ab} = \frac{1}{2}\bar{g}^{dc} (\nabla_a \delta g_{bc} + \nabla_b \delta g_{ca} - \nabla_c \delta g_{ab}), \quad (23)$$

note that $\gamma^d{}_{ab}$ is first order in δ . And Eq. (22) becomes

$$\delta R_{abc}{}^d \equiv \bar{R}_{abc}{}^d - R_{abc}{}^d = -\nabla_a \gamma^d{}_{bc} + \nabla_b \gamma^d{}_{ac}, \quad (24)$$

where the terms quadratic in γ^{bc} are second order and are dropped. The contraction of this last equation on b and d yields

$$\delta R_{ac} \equiv \bar{R}_{ac} - R_{ac} = -\nabla_a \gamma^b{}_{bc} + \nabla_b \gamma^b{}_{ac}, \quad (25)$$

After substitution from Eq. (23), this implies

$$-2\delta R_{ac} = \nabla^d \nabla_d \delta g_{ac} + \nabla_a \nabla_c \delta g - \nabla_a \nabla^b \delta g_{bc} - \nabla_c \nabla^b \delta g_{ba} + 2R^b{}_a{}^d{}_c \delta g_{bd} - 2R^b{}_{(a} \delta g_{c)b} \quad (26)$$

where $\delta g \equiv g^{ab} \delta g_{ab}$. This is equivalent to Eq.(35.58a) of MTW[1].

EXERCISE 2: Show that Eq. (26) follows from Eq. (25).

Solution: From Eq. (25)

$$\begin{aligned} -2\delta R_{ac} &= \nabla_a [g^{bd} (\nabla_c \delta g_{db} + \nabla_d \delta g_{cb} - \nabla_b \delta g_{cd})] - \nabla_b [g^{bd} (\nabla_a \delta g_{cd} + \nabla_c \delta g_{ad} - \nabla_d \delta g_{ac})] \\ &= \nabla_a \nabla_c (g^{bd} \delta g_{db}) - \nabla^b [\nabla_a \delta g_{cb} + \nabla_c \delta g_{ab} - \nabla_b \delta g_{ac}] \\ &= \nabla_a \nabla_c (g^{bd} \delta g_{db}) - \nabla^b \nabla_a \delta g_{cb} - \nabla^b \nabla_c \delta g_{ab} + \nabla^b \nabla_b \delta g_{ac} \\ &= \nabla^b \nabla_b \delta g_{ac} + \nabla_a \nabla_c (g^{bd} \delta g_{db}) - \nabla^b \nabla_a \delta g_{cb} - \nabla^b \nabla_c \delta g_{ab} \\ &= \nabla^b \nabla_b \delta g_{ac} + \nabla_a \nabla_c (g^{bd} \delta g_{db}) \\ &\quad - \nabla_a \nabla^b \delta g_{cb} - R^b{}_a{}^d{}_c \delta g_{db} - R^b{}_a{}^d{}_c \delta g_{dc} \\ &\quad - \nabla_c \nabla^b \delta g_{ab} - R^b{}_c{}^d{}_a \delta g_{db} - R^b{}_c{}^d{}_a \delta g_{da} \\ &= \nabla^b \nabla_b \delta g_{ac} + \nabla_a \nabla_c \delta g - \nabla_a \nabla^b \delta g_{cb} - \nabla_c \nabla^b \delta g_{ab} + 2R^b{}_a{}^d{}_c \delta g_{db} - 2R_{(a}{}^d \delta g_{c)d}. \end{aligned}$$

The fifth equality follows from the Ricci identity, and the sixth from symmetries of the Riemann tensor.

EXERCISE 3: Show that the contracted Bianchi identities follow automatically if the gravitational action is invariant under an infinitesimal coordinate transformation. HINT: follow a similar analysis as that leading up to Eq. (7).

V. CONCLUSION OF THE LAGRANGIAN FORMULATION OF GENERAL RELATIVITY

Now, we revisit the boundary integrals in the variation of the action of the gravitational field which is

$$\begin{aligned} \text{part of } \delta S_g &= \int_{4\text{-vol}} g^{ab} \delta R_{ab} \sqrt{-g} d^4x \\ &= \int_{4\text{-vol}} (-\nabla_a \gamma^b{}_{b^a} + \nabla_a \gamma^a{}_{b^b}) \sqrt{-g} d^4x \\ &= - \int_{4\text{-vol}} \partial_a (\sqrt{-g} \gamma^b{}_{b^a} - \sqrt{-g} \gamma^a{}_{b^b}) d^4x \\ &= - \oint_{3\text{-surf}} n_a (\gamma^b{}_{b^a} - \gamma^a{}_{b^b}) \sqrt{|h|} d^3x. \end{aligned} \quad (27)$$

We use Eq. (25) for the second equality, and versions of Gauss' law (cf. Eq. (75)) for third and fourth equalities. Thus the term which was dropped earlier leading up to Eq. (12) is,

indeed a surface integral. Let's look at this term a little more closely. γ^d_{ab} defined in Eq. (23) is an infinitesimal quantity, and the surface term is

$$\oint_{3\text{-surf}} -n_a (\gamma^b_{a} - \gamma^a_{b}) \sqrt{|h|} d^3x = - \oint_{3\text{-surf}} (n^a \nabla^b \delta g_{ab} - g^{bc} n^a \nabla_a \delta g_{bc}) \sqrt{|h|} d^3x \quad (28)$$

after γ^d_{ab} is removed in favor of δg_{ab} . A more interesting form for this surface term follows from a tedious analysis which is very similar to that between Eqs. (87) and (95). (To derive this expression from Eq. (95) make the following substitutions: $N \rightarrow 1$, $\bar{r} \rightarrow n^a$, $\sigma^{ab} \rightarrow h^{ab}$, $D \rightarrow \nabla$, $h_{ab} \rightarrow g_{ab}$, and $\delta h_{ab} \rightarrow \delta g_{ab}$.) The result is that the final surface term

$$- \oint_{3\text{-surf}} n^a h^{bc} (\nabla_b \delta g_{ca} - \nabla_a \delta g_{bc}) \sqrt{|h|} d^3x = - \oint_{3\text{-surf}} (\delta(h^{ab}) \nabla_a n_b + \delta(h^{ab}) h_{ab} \nabla_c n^c) \sqrt{|h|} d^3x + 2\epsilon \delta \left(\oint_{3\text{-surf}} h^{ab} K_{ab} \sqrt{|h|} d^3x \right). \quad (29)$$

Recall that $\epsilon = \pm 1$, depending upon whether n^a is spacelike or timelike, and that the extrinsic curvature of the boundary $K_{ab} = \epsilon h_a^c h_b^d \nabla_c n_d$. This expression shows directly how the boundary conditions and the action are related. If the action is modified to be

$$16\pi S = \int (R - \Lambda) \sqrt{g} d^4x - 2\epsilon \oint_{3\text{-surf}} h^{ab} K_{ab} \sqrt{|h|} d^3x, \quad (30)$$

then the variation of the additional boundary integral ultimately cancels out the ‘‘total’’ variation in Eq. (29). Only the first surface integral on the right hand side of Eq. (29) would remain. This modified action would then be appropriate for boundary conditions which specify just the 3-metric h_{ab} on the boundary, which would make the remaining surface term vanish. This choice is a version of Dirichlet boundary conditions. Note that this is a weaker condition than specifying the entire 4-metric on the boundary, which is often erroneously said to be required.

Also, recall that in a Lagrangian formulation of classical mechanics, the coordinates q_i are held fixed at the end points while the action is being extremized. The dynamical equations follow as a consequence. In that same spirit, it is apparent that the choice of Eq. (30) for the action implies that h_{ab} should be held fixed at the ‘‘end point’’, *i.e.* on the initial and final spacelike hypersurfaces. Thus, out of the entire metric, only h_{ab} is seen to be playing the role of the ‘‘canonical coordinates’’ q_i .

As an alternative to Eq. (30), we might have chosen

$$16\pi S = \int (R - \Lambda) \sqrt{g} d^4x - \epsilon \oint_{3\text{-surf}} h^{ab} K_{ab} \sqrt{|h|} d^3x \quad (31)$$

(Note that there is no factor of 2 in front of the boundary integral.) for the action, then the boundary integrals remaining after the variation are

$$\begin{aligned} & -\epsilon \oint_{3\text{-surf}} [\delta(h^{ab}) K_{ab} + \delta(h^{ab}) h_{ab} K_c^c] \sqrt{|h|} d^3x + \delta \left(\oint_{3\text{-surf}} h^{ab} K_{ab} \sqrt{|h|} d^3x \right) \\ & = \epsilon \oint_{3\text{-surf}} \left[-\delta(h^{ab}) K_{ab} - \delta(h^{ab}) h_{ab} K_c^c + \delta(h^{ab}) (K_{ab} - \frac{1}{2} h_{ab} K_c^c) + h^{ab} \delta K_{ab} \right] \sqrt{|h|} d^3x \\ & = \epsilon \oint_{3\text{-surf}} \left[-\frac{3}{2} \delta(h^{ab}) h_{ab} K_c^c + h^{ab} \delta K_{ab} \right] \sqrt{|h|} d^3x \\ & = \epsilon \oint_{3\text{-surf}} h^{ab} \delta (|h|^{3/2} K_{ab}) |h|^{-1} d^3x \end{aligned} \quad (32)$$

Thus, we see that Eq. (31) is the correct action for Neumann boundary conditions having $h^{3/2}K_{ab}$ be specified.

VI. MAXIMAL SURFACES AND THE VOLUME OF A HYPERSURFACE

This section should have been included in “Notes on Submanifolds.”

The volume (or area) of a bounded region of a hypersurface (or surface) is

$$A = \int_{\text{bnd region}} \sqrt{h} d^3x. \quad (33)$$

In this section we see how this volume changes when the hypersurface is slightly deformed.

Let the hypersurface be perpendicularly displaced by a small amount αn^a where α is assumed to be infinitesimal. Assume that the boundary of the hypersurface is undisturbed by this displacement. From the perspective of the hypersurface the small displacement appears as a small change δh_{ab} of the metric on the hypersurface,

$$\begin{aligned} \delta h_{ab} &= \mathcal{L}_{\alpha n^c} h_{ab} \\ &= \alpha n^c \nabla_c h_{ab} + h_{cb} \nabla_a (\alpha n^c) + h_{ac} \nabla_b (\alpha n^c) \\ &= \alpha (n^c \nabla_c h_{ab} + h_{cb} \nabla_a n^c + h_{ac} \nabla_b n^c) \\ &= \alpha \mathcal{L}_n h_{ab} = 2\epsilon K_{ab} \end{aligned} \quad (34)$$

as you showed in the exercise #6 in the “Notes on Submanifolds.”

Now, note that the change in the volume of the hypersurface is

$$\begin{aligned} \delta A &= \int \frac{1}{2\sqrt{h}} h^{ab} \delta h_{ab} d^3x \\ &= \int \frac{\alpha}{\sqrt{h}} h^{ab} K_{ab} d^3x \\ &= \int \frac{\alpha}{\sqrt{h}} K^a_a d^3x. \end{aligned} \quad (35)$$

A hypersurface is then said to be “maximal”, “minimal” or “extremal” (depending upon details) if the trace of the extrinsic curvature $K^a_a = 0$. And, from Eq. (52) in the notes on submanifolds, this condition is equivalent to having the divergence of the unit normal of the hypersurface be zero. A common example of a minimal surface is that formed by a soap film stretched across a wire frame — the surface minimizes the potential energy resulting from the tension in the soap film.

VII. HAMILTONIAN TREATMENT OF GENERAL RELATIVITY

In classical mechanics, the momentum canonically conjugate to a coordinate is defined by

$$p_i = \delta \mathcal{L} / \delta \dot{q}_i. \quad (36)$$

And, the Hamiltonian is a function of the p_i 's and q_i 's (NOT a function of the \dot{q}_i 's!)

$$H(p_i, q_i) = \sum_i p_i \dot{q}_i - \mathcal{L}. \quad (37)$$

The action principle now becomes

$$\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \delta(p\dot{q} - H) dt \\
&= \int_{t_1}^{t_2} \left(p\delta\dot{q} + \dot{q}\delta p - \delta p \frac{\delta H}{\delta p} - \delta q \frac{\delta H}{\delta q} \right) dt \\
&= \int_{t_1}^{t_2} \left[\frac{d}{dt}(p\delta q) - \dot{p}\delta q + \dot{q}\delta p - \delta p \frac{\delta H}{\delta p} - \delta q \frac{\delta H}{\delta q} \right] dt \\
&= [p\delta q]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[-\delta q \left(\dot{p} + \frac{\delta H}{\delta q} \right) + \delta p \left(\dot{q} - \frac{\delta H}{\delta p} \right) \right] dt
\end{aligned} \tag{38}$$

The action S is an extremum under an arbitrary change in path through phase space (with fixed endpoints) if and only if Hamilton's equations,

$$\dot{p} = -\frac{\delta H}{\delta q} \quad \text{and} \quad \dot{q} = \frac{\delta H}{\delta p} \tag{39}$$

are satisfied. These are the dynamical equations of classical mechanics.

A Hamiltonian approach to classical mechanics relies upon the use of a “time” function. This goes against the grain for general relativity, but the 3+1 formalism does an adequate job of treating gravity in a manner which is suitable for the Hamiltonian formalism. To find the Hamiltonian for gravity we start with (We are dropping the cosmological constant for my convenience in trying to get these notes correct.)

$$16\pi L = R \tag{40}$$

and rewrite it in terms of the 3+1 variables. One immediate complication is that R involves second order derivatives of the metric; hence, we use a more convenient description of R ,

$$R = 2n^a n^b \left(R_{ab} - \frac{1}{2} g_{ab} R \right) - 2n^a n^b R_{ab} \tag{41}$$

From the Hamiltonian constraint

$$2n^a n^b \left(R_{ab} - \frac{1}{2} g_{ab} R \right) = \mathcal{R} - K_{ab} K^{ab} + K^2, \tag{42}$$

and from the Ricci identity

$$\begin{aligned}
n^a R_{ac}{}^b n_b &= n^a (\nabla^c \nabla_a n_c - \nabla_a \nabla^c n_c) \\
&= \nabla^c (n^a \nabla_a n_c) - (\nabla^c n^a) \nabla_a n_c - \nabla_a (n^a \nabla^c n_c) + (\nabla_a n^a) \nabla_c n^c \\
&= \nabla^c (n^a \nabla_a n_c) - K_{ac} K^{ac} - \nabla_a (n^a \nabla^c n_c) + K^2;
\end{aligned} \tag{43}$$

also $\sqrt{g} = N\sqrt{h}$. Altogether,

$$\begin{aligned}
16\pi L = R &= \mathcal{R} - K_{ab} K^{ab} + K^2 - 2 \left[-K_{ac} K^{ac} + K^2 - \nabla_a (n^a \nabla^c n_c) + \nabla^c (n^a \nabla_a n_c) \right] \\
&= \mathcal{R} - K_{ab} K^{ab} + K^2 + 2K_{ac} K^{ac} - 2K^2 + 2\nabla_a (n^a \nabla^c n_c) - 2\nabla^c (n^a \nabla_a n_c) \\
&= \mathcal{R} + K_{ab} K^{ab} - K^2 + 2\nabla_a (n^a \nabla^c n_c) - 2\nabla^c (n^a \nabla_a n_c);
\end{aligned} \tag{44}$$

here \mathcal{R} and \mathcal{R}_{ab} are the scalar curvature and the Ricci tensor of constant time hypersurface whose metric is h_{ab} .

For a Hamiltonian treatment of General Relativity we must first identify the “coordinates.” From the Lagrangian formulation and with the machinery of the initial value formalism behind us, as a reasonable first try, we let the “coordinates” be the 3-metric h_{ab} of a spacelike hypersurface. h_{ab} along with the lapse N and shift vector β^a provide all ten components of the spacetime metric g_{ab} . In addition, we recall that the extrinsic curvature is related to the time derivative of h_{ab} by (note that $\epsilon = -1$)

$$\dot{h}_{ab} \equiv \mathcal{L}_t h_{ab} = -2NK_{ab} + \mathcal{L}_\beta h_{ab} = -2NK_{ab} + D_a \beta_b + D_b \beta_a \quad (45)$$

so that

$$K_{ab} = \frac{1}{2N} \left(-\dot{h}_{ab} + D_a \beta_b + D_b \beta_a \right). \quad (46)$$

Next, First we must find the “momenta” conjugate to the “coordinates.” Note that the time derivatives of the lapse, N , and shift vector β^a don’t appear in the Lagrangian, and these quantities have no conjugate momenta. This is consistent with the implication from the Lagrangian formulation that the “coordinates” should be h_{ab} and not include the lapse and shift. Below we see that the lapse N and shift β^a are Lagrange multipliers which enforce the constraint equations.

We are now prepared to find the “momenta” $P^{ab} \equiv \pi^{ab}/16\pi$ which are canonically conjugate to the “coordinates” h_{ab} .

$$P^{ab} = \frac{\delta \mathcal{L}}{\delta \dot{h}_{ab}} = \frac{\delta \mathcal{L}}{\delta K_{cd}} \frac{\delta K_{cd}}{\delta \dot{h}_{ab}} = -\frac{\delta \mathcal{L}}{\delta K_{cd}} \frac{1}{2N} \delta_c^a \delta_d^b = -\frac{1}{2N} \frac{\delta \mathcal{L}}{\delta K_{ab}}. \quad (47)$$

We ignore the boundary terms in determining the canonical momenta, as those are controlled by boundary conditions, not by dynamical equations.

EXERCISE 4: Perform the last step just described and show that

$$\pi^{ab} = 16\pi P^{ab} = -\sqrt{h}(K^{ab} - h^{ab}K). \quad (48)$$

Note that π^{ab} is not a tensor, but rather a tensor density because of the factor of \sqrt{h} . In the following I find it easiest to perform the usual tensor manipulations on the particular combination π^{ab}/\sqrt{h} which is a tensor.

It is now convenient to rewrite many of the 3+1 equations in terms of π^{ab} rather than in terms of K_{ab} . For starters:

$$K = \frac{\pi}{2\sqrt{h}}, \quad (49)$$

$$K^{ab} = -\frac{1}{\sqrt{h}} \left(\pi^{ab} - \frac{1}{2} h^{ab} \pi \right) \quad (50)$$

and

$$K^{ab} K_{ab} = \frac{1}{h} \left(\pi^{ab} \pi_{ab} - \pi^2 + \frac{3}{4} \pi^2 \right) = \frac{1}{h} \left(\pi^{ab} \pi_{ab} - \frac{1}{4} \pi^2 \right). \quad (51)$$

The Hamiltonian constraint is now

$$\mathcal{R} - K^{ab} K_{ab} + K^2 = \mathcal{R} - \frac{\pi_{ab} \pi^{ab}}{h} + \frac{\pi^2}{2h} \equiv \mathcal{N}, \quad (52)$$

and the momentum constraint is

$$D_b(K^{ab} - h^{ab}K) = -D_b(\pi^{ab}/\sqrt{h}) \equiv -\mathcal{B}^a. \quad (53)$$

A dynamical equation is

$$\begin{aligned} \mathcal{L}_t h_{ab} &= -2NK_{ab} + D_a\beta_b + D_b\beta_a \\ &= \frac{2N}{\sqrt{h}} \left(\pi^{ab} - \frac{1}{2}h^{ab}\pi \right) + D_a\beta_b + D_b\beta_a \equiv \mathcal{G}_{ab}. \end{aligned} \quad (54)$$

After a lengthy calculation, it may be verified that the ‘‘evolution equation,’’ from the Einstein equations is

$$\begin{aligned} \mathcal{L}_t \pi^{ab} &= -Nh^{1/2} \left(R^{ab} - \frac{1}{2}h^{ab}R \right) + \frac{1}{2}Nh^{-1/2}h^{ab} \left(\pi^{cd}\pi_{cd} - \frac{1}{2}\pi_c^c \pi_d^d \right) \\ &\quad - 2Nh^{-1/2} \left(\pi^{ac}\pi_c^b - \frac{1}{2}\pi_c^c \pi^{ab} \right) + h^{1/2} (D^a D^b N - h^{ab} D^c D_c N) \\ &\quad + h^{1/2} D_c (h^{-1/2} \pi^{ab} \beta^c) - \pi^{ac} D_c \beta^b - \pi^{bc} D_c \beta^a + 8\pi N h^{1/2} S^{ab} \\ &\equiv \mathcal{P}^{ab} + 8\pi N h^{1/2} S^{ab} \end{aligned} \quad (55)$$

for the time derivative of π^{ab} ; this also defines \mathcal{P}^{ab} for future use.

At this stage, the Lagrangian density along with the ‘‘coordinates’’ h_{ab} and their ‘‘momenta’’ π^{ab} have been identified. The next step is to determine the Hamiltonian from

$$16\pi \int_{\Sigma} \mathcal{H} d^3x = 16\pi \int_{\Sigma} \left(P^{ab} \dot{h}_{ab} - \mathcal{L} \right) d^3x. \quad (56)$$

The right hand side here is

$$\begin{aligned} &\int_{\Sigma} \left[\pi^{ab} \frac{2N}{\sqrt{h}} \left(\pi_{ab} - \frac{1}{2}\pi h_{ab} \right) + 2\pi^{ab} D_a \beta_b - (\mathcal{R} + K_{ab} K^{ab} - K^2) N \sqrt{h} \right] d^3x \\ &= \int_{\Sigma} \left[\pi^{ab} \frac{2N}{\sqrt{h}} \left(\pi_{ab} - \frac{1}{2}\pi h_{ab} \right) + 2\pi^{ab} D_a \beta_b - \left(\mathcal{R} + \frac{\pi_{ab} \pi^{ab}}{h} - \frac{\pi^2}{2h} \right) N \sqrt{h} \right] d^3x \\ &= \int_{\Sigma} \left[\left(-\mathcal{R} + \frac{\pi_{ab} \pi^{ab}}{h} - \frac{\pi^2}{2h} \right) N \sqrt{h} + 2\pi^{ab} D_a \beta_b \right] d^3x \end{aligned} \quad (57)$$

where we have used Eq. (44) (without the divergence terms) and Eqs. (49)-(53). Now we integrate the β^a term by parts

$$\int_{\Sigma} 2\pi^{ab} D_a \beta_b d^3x = \int_{\Sigma} 2D_a \left(\frac{\pi^{ab}}{\sqrt{h}} \beta_b \right) \sqrt{h} d^3x - \int_{\Sigma} 2\beta_b D_a \left(\frac{\pi^{ab}}{\sqrt{h}} \right) \sqrt{h} d^3x. \quad (58)$$

The first term on the right is a divergence term, may be written as a surface integral, and we will discard it for the time being—we consider all of the surface terms later. Finally we have for the Hamiltonian H

$$16\pi H = 16\pi \int_{\Sigma} \mathcal{H} d^3x = \int_{\Sigma} \left[\left(-\mathcal{R} + \frac{\pi_{ab} \pi^{ab}}{h} - \frac{\pi^2}{2h} \right) N \sqrt{h} - 2\beta_b D_a \left(\frac{\pi^{ab}}{\sqrt{h}} \right) \right] d^3x, \quad (59)$$

up to possible surface terms which we might wish to include.

Note that

$$16\pi \frac{\delta H}{\delta N} = \int_{\Sigma} \left(-\mathcal{R} + \frac{\pi_{ab}\pi^{ab}}{h} - \frac{\pi^2}{2h} \right) \delta N \sqrt{h} d^3x, \quad (60)$$

so that the Hamiltonian is an extremum with respect to arbitrary variations of the lapse δN , if and only if the Hamiltonian constraint is satisfied.

Note that

$$16\pi \frac{\delta H}{\delta \beta_a} = \int_{\Sigma} -2\delta\beta_b D_a \left(\frac{\pi^{ab}}{\sqrt{h}} \right) d^3x, \quad (61)$$

so that the Hamiltonian is an extremum with respect to arbitrary variations of the shift $\delta\beta_a$, if and only if the momentum constraint is satisfied.

VIII. SUMMARY OF CURRENT STATE

Our goal is to formulate a Hamiltonian approach to general relativity, which pays proper attention to “boundary terms”. At this point what is lacking is the treatment of these very boundary terms. It is most convenient, now, to define a first guess for H as in Eq. (59), then to derive the equations of motion while carefully tracking all boundary terms. At the end, we will be able to redefine H by consideration of these same boundary terms.

Before proceeding on this task, it is useful to consider how the geometry on a submanifold (a boundary, in our application) changes under a perturbation of the metric of the full manifold. This is covered in the following section.

IX. VARIATIONS OF GEOMETRICAL QUANTITIES IN 3+1 LANGAUGE

The boundary of Σ is assumed to be a two-dimensional surface upon which some given scalar field r is constant. The unit normal to the boundary is \bar{r}^a , and the 2-metric of the boundary is σ_{ab} with determinant σ . The function ρ (defined below) plays the role of the “lapse function” for this boundary two dimensional boundary as embedded in the three dimensional Σ .

The variation of the Hamiltonian is manipulated using the following conventions:

$$\delta h_{ab} \equiv \delta(h_{ab}), \quad (62)$$

$$\delta h^{ab} \equiv h^{ac} h^{bd} \delta h_{cd} = -\delta(h^{ab}), \quad (63)$$

$$h \equiv \det(h_{ab}), \quad (64)$$

$$\text{and } \delta h/h = h^{ab} \delta h_{ab} = \delta h_a^a. \quad (65)$$

The unit normal to a surface defined by a scalar field $r = \text{constant}$ is

$$\bar{r}_a \equiv \rho D_a r, \quad (66)$$

where

$$\rho^{-2} \equiv h^{ab} D_a r D_b r. \quad (67)$$

Here r is not necessarily related to the radial coordinate of flat space. The metric of the boundary, as well as the projection operator onto the boundary is

$$\sigma^{ab} \equiv h^{ab} - \bar{r}^a \bar{r}^b \quad (68)$$

and the determinant of σ_{ab} is σ . Also,

$$D_a \bar{r}_b = \sigma_{(a}{}^c \sigma_b)^d D_c \bar{r}_d - \rho^{-1} \bar{r}_a \sigma_b{}^d D_d \rho. \quad (69)$$

Variations of the geometry at the boundary, with the location of the boundary fixed, obey

$$\delta(\bar{r}_a) = \delta\rho D_a r = \bar{r}_a \delta\rho/\rho, \quad (70)$$

and

$$\bar{r}^a \delta(\bar{r}_a) = -\delta(\bar{r}^a) \bar{r}_a = \delta\rho/\rho \quad (71)$$

so that

$$\delta\bar{r}^a \equiv \delta(\bar{r}^a) = \sigma_b{}^a \delta\bar{r}^b - \bar{r}^a \delta\rho/\rho. \quad (72)$$

Now from $\delta(\sigma^{ab} \bar{r}_b) = 0$ and Eq. (70) it follows that

$$\delta(\sigma^{ab}) \bar{r}_b = 0. \quad (73)$$

And the variation in h^{ab} may be related to the variation of σ^{ab} via Eq. (68),

$$\delta(h^{ab}) = \delta(\sigma^{ab}) + \bar{r}^a \sigma_c{}^b \delta\bar{r}^c + \bar{r}^b \sigma_c{}^a \delta\bar{r}^c - 2\bar{r}^a \bar{r}^b \delta\rho/\rho. \quad (74)$$

Before starting on the variation of the Hamiltonian it is useful to note a variety of forms of Gauss' law:

$$\begin{aligned} \int (D_b A^b) h^{1/2} d^3x &= \int \frac{\partial}{\partial x^b} (h^{1/2} A^b) d^3x = \oint (D_b r) A^b h^{1/2} d^2x \\ &= \oint \frac{\bar{r}_b A^b}{\rho} h^{1/2} d^2x = \oint \bar{r}_b A^b \sigma^{1/2} d^2x \end{aligned} \quad (75)$$

where we use Eq. (66) and

$$h = \sigma \rho^2. \quad (76)$$

X. THE VARIATION FINALLY BEGINS

The starting point of the Hamiltonian formalism is the definition

$$16\pi H_0 \equiv - \int \left\{ N \left[\mathcal{R} + h^{-1} \left(\frac{1}{2} \pi_a{}^a \pi_b{}^b - \pi^{ab} \pi_{ab} \right) \right] + 2\beta^a D_b (h^{-1/2} \pi_a{}^b) \right\} h^{1/2} d^3x. \quad (77)$$

With $\mathcal{R} = h^{ab} \mathcal{R}_{ab}$,

$$\begin{aligned} \delta\mathcal{R} &= \delta(h^{ab}) \mathcal{R}_{ab} + h^{ab} \delta\mathcal{R}_{ab} = -\delta h_{ab} \mathcal{R}^{ab} + h^{ab} \delta\mathcal{R}_{ab} \\ &= -\delta h_{ab} \mathcal{R}^{ab} + h^{ab} (D^c D_{(a} \delta h_{b)c} - \frac{1}{2} D_a D_b \delta h_c{}^c - \frac{1}{2} D^c D_c \delta h_{ab}) \\ &= -\delta h_{ab} \mathcal{R}^{ab} + D^a D^b \delta h_{ab} - D^a D_a \delta h_c{}^c \end{aligned} \quad (78)$$

the second equality follows from an intermediate step in the derivation of Eq. (26), above, which you came across in your solution to Exercise 2.

The variations of the different parts of the volume integral are

$$\begin{aligned}
h^{-1/2}\delta(N\mathcal{R}h^{1/2}) &= \delta N \mathcal{R} - N \delta h_{ab}(\mathcal{R}^{ab} - \frac{1}{2}h^{ab}\mathcal{R}) + \delta h_{ab}(D^a D^b N - h^{ab}D^c D_c N) \\
&+ D^a(ND^b \delta h_{ab}) - D^a(ND_a \delta h_b^b) \\
&- D^a(\delta h_{ab}D^b N) + D^a(\delta h_b^b D_a N), \tag{79}
\end{aligned}$$

$$\begin{aligned}
h^{-1/2}\delta\left[\frac{N}{h}\left(\frac{1}{2}\pi_a^a \pi_b^b - \pi^{ab}\pi_{ab}\right)h^{1/2}\right] &= \frac{\delta N}{h}\left(\frac{1}{2}\pi_a^a \pi_b^b - \pi^{ab}\pi_{ab}\right) - \frac{N\delta h_c^c}{2h}\left(\frac{1}{2}\pi_a^a \pi_b^b - \pi^{ab}\pi_{ab}\right) \\
&+ \frac{N}{h}\left(\pi_c^c \pi^{ab}\delta h_{ab} - 2\pi^{ac}\pi_c^b \delta h_{ab}\right) \\
&+ \frac{N}{h}\left(\delta\pi^{ab}h_{ab}\pi_c^c - 2\delta\pi^{ab}\pi_{ab}\right), \tag{80}
\end{aligned}$$

and

$$\begin{aligned}
h^{-1/2}\delta\left[2\beta^a D_b(\pi_a^b/h^{1/2})h^{1/2}\right] &= 2\delta\beta^a D_b(\pi_a^b/h^{1/2}) - 2\delta\pi^{ab}D_a\beta_b/h^{1/2} \\
&- \delta h_{ab}\left[2\pi^{bc}D_c\beta^a/h^{1/2} - D_c(\beta^c\pi^{ab}/h^{1/2})\right] \\
&- D_a(\beta^a\pi^{bc}\delta h_{bc}/h^{1/2}) \\
&+ D_a(2\beta_b\delta\pi^{ab}/h^{1/2}) + D_a(2\beta^b\pi^{ac}\delta h_{bc}/h^{1/2}). \tag{81}
\end{aligned}$$

After the substitution of these expressions into δH_0

$$\begin{aligned}
16\pi\delta H_0 &= -\int(\delta N\mathcal{N}h^{1/2} + 2\delta\beta^a\mathcal{B}_a h^{1/2} + \delta h_{ab}\mathcal{P}^{ab} - \delta\pi^{ab}\mathcal{G}_{ab})d^3x \\
&+ \int D_a(\beta^a\pi^{bc}\delta h_{bc}/h^{1/2} - 2\beta_b\delta\pi^{ab}/h^{1/2} - 2\beta^b\pi^{ac}\delta h_{bc}/h^{1/2})h^{1/2}d^3x \\
&+ \int[-D^a(ND^b\delta h_{ab}) + D^a(ND_a\delta h_b^b) + D^a(\delta h_{ab}D^b N) - D^a(\delta h_b^b D_a N)]h^{1/2}d^3x; \tag{82}
\end{aligned}$$

recall the definitions of \mathcal{N} , \mathcal{B}_a , \mathcal{P}^{ab} and \mathcal{G}_{ab} in Eqs. (52)-(55). After the application of Gauss' law, the variation of H_0 becomes

$$\begin{aligned}
16\pi\delta H_0 &= -\int(\delta N\mathcal{N}h^{1/2} + 2\delta\beta^a\mathcal{B}_a h^{1/2} + \delta h_{ab}\mathcal{P}^{ab} - \delta\pi^{ab}\mathcal{G}_{ab})d^3x \\
&+ \oint(\bar{r}_a\beta^a\pi^{bc}\delta h_{bc}/h^{1/2} - 2\bar{r}_a\beta_b\delta\pi^{ab}/h^{1/2} - 2\bar{r}_a\beta^b\pi^{ac}\delta h_{bc}/h^{1/2})\sigma^{1/2}d^2x \\
&+ \oint(-N\bar{r}^a D^b\delta h_{ab} + N\bar{r}^a D_a\delta h_b^b + \bar{r}^a\delta h_{ab}D^b N - \bar{r}^a\delta h_b^b D_a N)\sigma^{1/2}d^2x. \tag{83}
\end{aligned}$$

From the Lagrangian formulation of general relativity, we expect that δH_0 should be expressible in a form where each surface integral is either a total variation or consists of terms containing only the variations of the 3-metric of the boundary. Such terms contain δN , $\delta\beta^a$ or $\delta\sigma^{ab}$, but do not contain derivatives of these quantities. We proceed to manipulate the surface integrals of Eq. (83) into precisely this form.

The terms in the surface integrals on the boundary which involve π^{ab} are easily rewritten as

$$\begin{aligned}
16\pi\delta H_0^{\pi\partial} &\equiv \oint (\bar{r}_a\beta^a\pi^{bc}\delta h_{bc}/h^{1/2} - 2\bar{r}_a\beta_b\delta\pi^{ab}/h^{1/2} - 2\bar{r}_a\beta^b\pi^{ac}\delta h_{bc}/h^{1/2})\sigma^{1/2}d^2x \\
&= \oint [\bar{r}_a\beta^a\pi^{bc}\delta h_{bc}/h^{1/2} + 2\bar{r}_a\delta\beta^b\pi_b^a/h^{1/2}]\sigma^{1/2}d^2x \\
&\quad - 2\delta \left[\oint (\bar{r}_a\beta^b\pi_b^a/h^{1/2})\sigma^{1/2}d^2x \right]. \tag{84}
\end{aligned}$$

This follows from

$$\begin{aligned}
2\delta \oint (\bar{r}_a\beta^b\pi_b^a/h^{1/2})\sigma^{1/2}d^2x &= 2\delta \oint \beta^b\pi^{ac}h_{bc}D_a r d^2x \\
&= \oint (2\bar{r}_a\delta\beta^b\pi_b^a/h^{1/2} + 2\bar{r}_a\beta_b\delta\pi^{ab}/h^{1/2} + 2\bar{r}_a\beta^b\pi^{ac}\delta h_{bc}/h^{1/2})\sigma^{1/2}d^2x, \tag{85}
\end{aligned}$$

where the first equality depends upon Eq. (76).

The terms involving δh_{ab} present more of a challenge.

$$16\pi\delta H_0^{h\partial} \equiv \oint (-N\bar{r}^a D^b \delta h_{ab} + N\bar{r}^a D_a \delta h_b^b + \bar{r}^a \delta h_{ab} D^b N - \bar{r}^a \delta h_b^b D_a N)\sigma^{1/2}d^2x. \tag{86}$$

The first two terms of the integral in Eq. (86) are

$$\begin{aligned}
&\oint [-N\bar{r}^a h^{bc} D_b \delta h_{ac} + N\bar{r}^a h^{bc} D_a \delta h_{bc}]\sigma^{1/2}d^2x \\
&= \oint [N\bar{r}_a \sigma_c^b D_b \delta(h^{ac}) - N\bar{r}^a \sigma_{bc} D_a \delta(h^{bc})]\sigma^{1/2}d^2x. \tag{87}
\end{aligned}$$

The first term on the right hand side of Eq. (87) gives

$$\begin{aligned}
\oint N\bar{r}_a \sigma_c^b D_b \delta(h^{ac})\sigma^{1/2}d^2x &= \oint N\bar{r}_a \sigma_c^b D_b [\delta(\sigma^{ac}) + \bar{r}^a \sigma_d^c \delta\bar{r}^d + \bar{r}^c \sigma_d^a \delta\bar{r}^d - 2\bar{r}^a \bar{r}^c \delta\rho/\rho]\sigma^{1/2}d^2x \\
&= \oint N\bar{r}_a \sigma_c^b D_b [\delta(\sigma^{ac}) + \bar{r}^a \sigma_d^c \delta\bar{r}^d - 2\bar{r}^a \bar{r}^c \delta\rho/\rho]\sigma^{1/2}d^2x \\
&= \oint N[-\sigma_c^b \delta(\sigma^{ac}) D_b \bar{r}_a + \bar{D}_c(\sigma^c{}_a \delta\bar{r}^d) - 2\rho^{-1} \delta\rho \sigma^{bc} D_b \bar{r}_c]\sigma^{1/2}d^2x \tag{88}
\end{aligned}$$

where \bar{D}_b is the two dimensional derivative operator on the boundary, and we made use of Eq. (74). Now, $N\bar{D}_c(\sigma^c{}_a \delta\bar{r}^d) = \bar{D}_c(N\sigma^c{}_a \delta\bar{r}^d) - \sigma^c{}_a \delta\bar{r}^d \bar{D}_c N$ and the divergence here contributes to an integral of a two dimensional divergence over the two dimensional boundary which has no boundary, itself. The result is zero from Stoke's theorem. Finally, the first term in the integral on the right hand side of Eq. (87) is reduced to

$$\oint N\bar{r}_a \sigma_c^b D_b \delta(h^{ac})\sigma^{1/2}d^2x = \oint [-N\delta(\sigma^{ab})D_a \bar{r}_b - \sigma^a{}_b \delta\bar{r}^b D_a N - 2N\rho^{-1} \delta\rho \sigma^{bc} D_b \bar{r}_c]\sigma^{1/2}d^2x. \tag{89}$$

The second term of the right hand side of Eq. (87) is

$$\begin{aligned}
& - \oint N \bar{r}^a \sigma_{bc} D_a \delta(h^{bc}) \sigma^{1/2} d^2x = - \oint N \bar{r}^a \sigma_{bc} D_a [\delta(\sigma^{bc}) + \bar{r}^c \sigma_d^b \delta \bar{r}^d + \bar{r}^b \sigma_d^c \delta \bar{r}^d - 2\bar{r}^b \bar{r}^c \delta \rho / \rho] \sigma^{1/2} d^2x \\
& = - \oint N \bar{r}^a \sigma_{bc} D_a [\delta(\sigma^{bc}) + 2\bar{r}^c \sigma_d^b \delta \bar{r}^d] \sigma^{1/2} d^2x \\
& = - \oint N \{ \bar{r}^a D_a [\sigma_{bc} \delta(\sigma^{bc})] - \bar{r}^a \delta(\sigma^{bc}) D_a (\bar{r}_b \bar{r}_c) + 2\bar{r}^a \sigma_b^c D_a (\bar{r}_c \sigma_d^b \delta \bar{r}^d) \} \sigma^{1/2} d^2x \\
& = - \oint N \{ \bar{r}^a D_a [\sigma_{bc} \delta(\sigma^{bc})] + 2\bar{r}^a \sigma_d^c \delta \bar{r}^d D_a \bar{r}_c \} \sigma^{1/2} d^2x. \tag{90}
\end{aligned}$$

Now focus on

$$\begin{aligned}
2\delta(\sigma^{ab} D_a n_b) &= 2\delta \left[\frac{1}{h^{1/2}} \frac{\partial (h^{1/2} \bar{r}^a)}{\partial x^a} \right] \\
&= -\frac{\delta h}{h} D_a \bar{r}^a + D_a \left(\frac{\delta h}{h} \bar{r}^a \right) + 2D_a \delta \bar{r}^a \\
&= -\delta h_b^b D_a \bar{r}^a + D_a (\delta h_b^b \bar{r}^a + 2\delta \bar{r}^a) \\
&= -\delta h_b^b D_a \bar{r}^a + D_a (\delta h_b^b \bar{r}^a + 2\sigma_b^a \delta \bar{r}^b - 2\bar{r}^a \delta \rho / \rho) \\
&= -\delta h_b^b D_a \bar{r}^a + D_a \left[\frac{\delta (h/\rho^2)}{h/\rho^2} \bar{r}^a + 2\sigma_b^a \delta \bar{r}^b \right] \\
&= -\delta h_b^b D_a \bar{r}^a + D_a \left(\frac{\delta \sigma}{\sigma} \bar{r}^a + 2\sigma_b^a \delta \bar{r}^b \right) \\
&= -h^{-1} \delta h D_a \bar{r}^a - D_a [\sigma_{bc} \delta(\sigma^{bc}) \bar{r}^a - 2\sigma_b^a \delta \bar{r}^b]. \tag{91}
\end{aligned}$$

Substitute this into the result of Eq. (90) to obtain

$$\begin{aligned}
& - \oint N \bar{r}^a \sigma_{bc} D_a \delta(h^{bc}) \sigma^{1/2} d^2x \\
& = - \oint N \{ \bar{r}^a D_a [\sigma_{bc} \delta(\sigma^{bc})] + 2\bar{r}^a \sigma_d^c \delta \bar{r}^d D_a \bar{r}_c \\
& \quad - 2\delta(\sigma^{ab} D_a n_b) - h^{-1} \delta h D_c \bar{r}^c - D_a [\sigma_{bc} \delta(\sigma^{bc}) \bar{r}^a] + 2D_a (\sigma_b^a \delta \bar{r}^b) \} \sigma^{1/2} d^2x \\
& = \oint N [2\delta(\sigma^{ab} D_a \bar{r}_b) + h^{-1} \delta h D_c \bar{r}^c + \sigma_{bc} \delta(\sigma^{bc}) D_a \bar{r}^a - 2\sigma_c^a D_a (\sigma^c_b \delta \bar{r}^b)] \sigma^{1/2} d^2x \\
& = \oint N [2\delta(\sigma^{ab} D_a \bar{r}_b) + h^{-1} \delta h D_c \bar{r}^c - \sigma \delta \sigma D_a \bar{r}^a - 2\sigma_c^a D_a (\sigma^c_b \delta \bar{r}^b)] \sigma^{1/2} d^2x \\
& = \oint N [2\delta(\sigma^{ab} D_a \bar{r}_b) + 2\rho^{-1} \delta \rho D_a \bar{r}^a - 2\sigma_c^a D_a (\sigma^c_b \delta \bar{r}^b)] \sigma^{1/2} d^2x. \tag{92}
\end{aligned}$$

But,

$$\begin{aligned}
\oint N [-2\sigma_c^a D_a (\sigma^c_b \delta \bar{r}^b)] \sigma^{1/2} d^2x &= \oint [-2\bar{D}_a (N \sigma^a_b \delta \bar{r}^b) + 2\sigma^a_b \delta \bar{r}^b D_a N] \sigma^{1/2} d^2x \\
&= \oint [2\sigma^a_b \delta \bar{r}^b D_a N] \sigma^{1/2} d^2x \tag{93}
\end{aligned}$$

from Stoke's Theorem again using the fact that a boundary has no boundary, itself. Finally

$$\begin{aligned}
& - \oint N \bar{r}^a \sigma_{bc} D_a \delta(h^{bc}) \sigma^{1/2} d^2 x \\
& = \oint [2N \delta(\sigma^{ab} D_a \bar{r}_b) + 2N \rho^{-1} \delta \rho D_a \bar{r}^a + 2\sigma^a{}_b \delta \bar{r}^b D_a N] \sigma^{1/2} d^2 x \\
& = 2\delta \left(\oint N D_a \bar{r}^a \sigma^{1/2} d^2 x \right) \\
& \quad + \oint [(-N \sigma^{-1} \delta \sigma - 2\delta N + 2N \rho^{-1} \delta \rho) D_a \bar{r}^a + 2\sigma^a{}_b \delta \bar{r}^b D_a N] \sigma^{1/2} d^2 x. \tag{94}
\end{aligned}$$

Now substituting Eqs. (89) and (94) into Eq. (87) while using Eq. (76), we have

$$\begin{aligned}
\oint N [-\bar{r}^a D^b \delta h_{ab} + \bar{r}^a D_a \delta h_b{}^b] \sigma^{1/2} d^2 x & = \oint [-N \delta(\sigma^{ab}) D_a \bar{r}_b - N \sigma^{-1} \delta \sigma D_a \bar{r}^a \\
& \quad + \sigma^a{}_b \delta \bar{r}^b D_a N - 2\delta N D_a \bar{r}^a] + 2\delta \left(\oint N D_a \bar{r}^a d^2 x \right) \tag{95}
\end{aligned}$$

Also using Eq. (74) note that

$$\begin{aligned}
& \oint (\sigma^a{}_b \delta \bar{r}^b D_a N + \bar{r}^a \delta h_{ab} D^b N - h^{-1} \delta h \bar{r}^a D_a N) \sigma^{1/2} d^2 x \\
& = \oint \{ \sigma^a{}_b \delta \bar{r}^b D_a N - \bar{r}_a [\delta(\sigma^{ab}) + \bar{r}^a \sigma^b{}_c \delta \bar{r}^c + \bar{r}^b \sigma^a{}_c \delta \bar{r}^c \\
& \quad - 2\bar{r}^a \bar{r}^b \rho^{-1} \delta \rho] D_b N - (2\rho^{-1} \delta \rho + \sigma^{-1} \delta \sigma) \bar{r}^a D_a N \} \sigma^{1/2} d^2 x \\
& = - \oint \sigma^{-1} \delta \sigma \bar{r}^a D_a N \sigma^{1/2} d^2 x. \tag{96}
\end{aligned}$$

After substitution Eqs. (89) and (94) into Eq. (87) and use of Eqs. (76) and (96), the δh surface terms of $16\pi\delta H_0$ in Eq. (83) reduce to

$$\begin{aligned}
16\pi\delta H_0^{h\partial} & = 2\delta \left(\oint N D_a \bar{r}^a \sigma^{1/2} d^2 x \right) \\
& \quad - \oint [N \delta(\sigma^{ab}) D_a \bar{r}_b + (2\delta N + N \sigma^{-1} \delta \sigma) D_a \bar{r}^a + \sigma^{-1} \delta \sigma \bar{r}^a D_a N] h^{1/2} d^3 x. \tag{97}
\end{aligned}$$

Using Eq. (83) with Eqs. (84) and (97) we define

$$16\pi H_1 = 16\pi H_0 + \oint 2\bar{r}^a \beta^b h^{-1/2} \pi_{ab} \sigma^{1/2} d^2 x - \oint 2N D_a \bar{r}^a \sigma^{1/2} d^2 x, \tag{98}$$

and the variation of H_1 results in Eq. (101), below.

XI. CONCLUSIONS

At the end of the previous section we defined

$$\begin{aligned}
16\pi H_1 & \equiv - \int (N\mathcal{N} + 2\beta^a \mathcal{B}_a) h^{1/2} d^3 x \\
& \quad + \oint 2n^a \beta^b h^{-1/2} \pi_{ab} \sigma^{1/2} d^2 x - \oint 2N D_a n^a \sigma^{1/2} d^2 x \tag{99}
\end{aligned}$$

where the volume integral is over Σ_t . The vector n_a is the outward pointing unit normal to the bounding two-surface, σ_{ab} is both the metric of this two-surface and the projection operator onto the two-surface,

$$\sigma_{ab} \equiv h_{ab} - n_a n_b, \quad (100)$$

σ is its determinant and $-D_a n^a$ is the trace of the extrinsic curvature of the boundary as embedded in Σ_t .

An arbitrary, infinitesimal variation of N , β^a , h_{ab} and π^{ab} , which holds the location of the boundary fixed, results in an infinitesimal change in H_1

$$\begin{aligned} 16\pi\delta H_1 = & - \int (\delta N \mathcal{N} h^{1/2} + 2\delta\beta^a \mathcal{B}_a h^{1/2} + \delta h_{ab} \mathcal{P}^{ab} - \delta\pi^{ab} \mathcal{G}_{ab}) d^3x \\ & - \oint \{ \delta\sigma^{ab} [N D_a n_b - \sigma_{ab} D_c (N n^c)] + 2\delta N D_a n^a \} d^2x \\ & + \oint (n_a \beta^a h^{-1/2} \pi^{bc} \delta h_{bc} + 2n^a \delta\beta^b h^{-1/2} \pi_{ab}) \sigma^{1/2} d^2x. \end{aligned} \quad (101)$$

In our analysis, at least one connected component of the boundary is usually in a weak-field zone. And we assume that this component is spherically symmetric with constant Cartesian radius r . The outward-pointing unit (flat space) vector, normal to a surface of constant r , is $n_i^0 \equiv \nabla_i^0 r$ and should not be confused with n_a which is orthogonal to the 2-surface that bounds Σ_t and is normalized with h_{ab} . The metric of the constant- r 2-sphere embedded in flat space is σ_{ij}^0 where

$$\sigma_{ij}^0 = r \nabla_i^0 n_j^0 \quad (102)$$

and the determinant of σ_{ij}^0 is σ_0 , the 0 distinguishes this from the determinant of σ_{ab} , the actual metric of the 2-surface that bounds Σ_t . The symbol r used as a tensor index on one of the Cartesian tensors denotes the implied contraction of the index with n_0^i and $\partial_r = n_0^i \nabla_i^0$

We assume that at a large distance the shift vector might have an axial component and may be written as

$$\beta^i = \Omega \Phi^i + S^i, \quad (103)$$

where S^j falls off sufficiently fast, and where $\Phi^i \partial / \partial x^i \equiv \partial / \partial \phi$.

The first surface integral of Eq. (101) may be rewritten as

$$\begin{aligned} & - \oint_{r_0} \{ \delta\sigma^{ab} [N D_a n_b - \sigma_{ab} D_c (N n^c)] + 2\delta N D_a n^a \} d^2x \\ & = -\delta \left(\oint_{r_0} N^2 \nabla_i^0 n_0^i \sigma^{1/2} d^2x \right) \\ & \quad - \oint_{r_0} \left[\delta(\sigma^{1/2} \sigma^{ab}) N (D_a n_b - \frac{1}{2} \sigma_{ab} D_c n^c) + \delta(\sigma^{1/2} N^2) (N^{-1} D_a n^a - \nabla_i^0 n_0^i) \right. \\ & \quad \left. + 2\delta(\sigma^{1/2}) n^a D_a N \right] d^2x. \end{aligned} \quad (104)$$

The total variation of the surface integral on the right hand side will be absorbed into yet another definition for H .

The term involving $n^a \delta \beta^b \pi_{ab}$ in Eq. (101) is

$$\begin{aligned} \oint_{r_0} 2n^a \delta \beta^b h^{-1/2} \pi_{ab} \sigma^{1/2} d^2x &= \delta \left(2\Omega \oint_{r_0} n^a \Phi^b h^{-1/2} \pi_{ab} \sigma^{1/2} d^2x \right) + 16\pi\Omega \delta J \\ &+ \oint_{r_0} 2n^a \delta S^b h^{-1/2} \pi_{ab} \sigma^{1/2} d^2x, \end{aligned} \quad (105)$$

where

$$16\pi J \equiv - \oint_{r_0} 2n^a \Phi^b h^{-1/2} \pi_{ab} \sigma^{1/2} d^2x; \quad (106)$$

with the boundary in the weak-field region J is the angular momentum. Again, the surface integral of the total variation on the right hand side here will be absorbed in a new definition for H .

The new version of H including the total variations from Eqs. (104) and (105) is

$$\begin{aligned} 16\pi H_2 &\equiv - \int (N\mathcal{N} + 2\beta^a \mathcal{B}_a) h^{1/2} d^3x \\ &- \oint_{r_0} N (2D_a n^a - N \nabla_i^0 n_0^i) \sigma^{1/2} d^2x + \oint_{r_0} \nabla_i^0 n_0^i \sigma_0^{1/2} d^2x \\ &+ \oint_{r_0} 2n^a S^b h^{-1/2} \pi_{ab} \sigma^{1/2} d^2x. \end{aligned} \quad (107)$$

The second surface integral is a constant $8\pi r_0$ that does not change under a variation but is included so that the surface integrals of H_2 evaluate to M . The last surface integral vanishes as $r_0 \rightarrow \infty$.

An arbitrary, infinitesimal variation of N , β^a , h_{ab} and π^{ab} results in an infinitesimal change of H_2 , and

$$\begin{aligned} 16\pi \delta H_2 &= - \int [\text{E eqns}] d^3x + 16\pi\Omega \delta J \\ &- \oint_{r_0} \left[\delta (\sigma^{1/2} \sigma^{ab}) N \left(D_a n_b - \frac{1}{2} \sigma_{ab} D_c n^c \right) + \delta (\sigma^{1/2} N^2) (N^{-1} D_a n^a - \nabla_i^0 n_0^i) \right. \\ &\quad \left. + 2\delta (\sigma^{1/2}) n^a D_a N \right] d^2x \\ &+ \oint_{r_0} [n_a \beta^a h^{-1/2} \pi^{bc} \delta h_{bc} + 2n^a \delta S^b h^{-1/2} \pi_{ab}] \sigma^{1/2} d^2x. \end{aligned} \quad (108)$$

The symbol [E eqns] is an abbreviation for the first integrand in Eq. (101) which gives both the constraint equations and the dynamical Einstein equations. Each of the remaining surface integrals vanish as $r_0 \rightarrow 0$ for an asymptotically flat metric.

The ‘‘value’’ of H when the field equations are satisfied gives the mass of the system, as measured at infinity, in terms of boundary integrals:

$$16\pi M \equiv - \oint_{r_0} N (2D_a n^a - N \nabla_i^0 n_0^i) \sigma^{1/2} d^2x + \oint_{r_0} \nabla_i^0 n_0^i \sigma_0^{1/2} d^2x \quad (109)$$

and if N is approximately a constant $N_0 \neq 1$ in the weak field region then N should be replaced by N/N_0 in the two places it occurs.

You might note that the form for H_2 is rather more complicated than that given by, say, Wald [2], or Poisson [3]. Our form for the Hamiltonian allows for evaluation under modestly weaker asymptotic requirements than that of the others. But, this is not a substantial improvement.

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