A charged particle in free-fall near the Earth is accelerating and must radiate as a consequence of Maxwell’s equations. However, the principle of equivalence implies that an observer falling with the particle sees no acceleration and, therefore, no radiation. The apparent contradiction is well understood, but recent analysis shines new light on this old paradox.
Outline

• Brief discussion of the Principle of Equivalence

• Classical descriptions of radiation reaction.

• An apparent paradox involving radiation reaction and the principle of equivalence.

• The resolution of the paradox.

• A sketch of some details.

• Implications for the principle of equivalence.
Einstein’s Principle of Equivalence

Uniform gravitational field = uniformly accelerating reference frame

Uniform gravitational field − uniformly accelerating reference frame = no gravitational field

⇒ A small object at rest remains at rest.
⇒ A small object moves along a geodesic of spacetime, requires that $R_{\text{obj}} \ll L$, where $L$ is a length scale of the background.
In relativity, acceleration measures the deviation from geodesic motion.

Use dimensional analysis to show that this also requires $m_{\text{obj}} \ll L$

Perturbation in gravitational field = $h \sim m/L$

$m\ddot{a} = d\vec{P}/dt \sim h^2 \sim m^2/L^2$ from radiation reaction

$\ddot{a} \sim m/L^2$

and $\ddot{a}$ is small if $m \ll L$
In classical electricity and magnetism there are two common types of problems:

**Given q’s and i’s, find E’s and B’s**

\[ \nabla \cdot \mathbf{E} = 4\pi \rho \quad \nabla \times \mathbf{B} = 4\pi \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \]

**Given E’s and B’s find the motion of q’s and i’s.**

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]

It is implicit that a point charge does not interact with its own field... until one faces radiation reaction or the “self force.”

Then, just what is the “self” field, the particle’s own field?
First focus on radiation reaction.

- What does Jackson say about radiation reaction?

\[ F_{\text{ext}} + F_{\text{rad}} = m\dot{v} \]

\[ F_{\text{rad}} = \frac{2}{3} \frac{e^2}{c^3} \ddot{v} = \frac{2}{3} \frac{e^2}{mc^3} F_{\text{ext}} \quad \text{Abraham-Lorentz force} \]

- What did Dirac say about radiation reaction?

\[ E_{\text{ret}} = \frac{1}{2} (E_{\text{ret}} + E_{\text{adv}}) + \frac{1}{2} (E_{\text{ret}} - E_{\text{adv}}) \]

\[ E_{\text{ret}} = \frac{1}{2} (E_{\text{ret}} + E_{\text{adv}}) + E_{\text{rad}} \quad \text{(defines } E_{\text{rad}}) \]

Coulomb
inhomogeneous
singular
radiation
homogeneous
smooth

Dirac calculated the flux of energy and momentum across a sphere enclosing the charge and deduced that radiation reaction arises from the interaction of the charge with \( E_{\text{rad}} \).
The equivalence principle creates an apparent paradox
Does a charged particle, falling toward the Earth, radiate?

Yes:

The particle accelerates.

Maxwell’s equations \( \Rightarrow \) radiation.

No:

Equivalence principle: A particle in free-fall is not accelerating, from a local point of view.

A local observer must see no radiation.
A simple resolution of the paradox

Both descriptions are correct.

- Radiation is a global phenomenon not a local one.
- A distant observer sees radiation.
- A local observer sees no radiation.
- Radiation comes from the distortion of \( E \)'s and \( B \)'s by the curvature of spacetime.
The paradox reappears when we consider radiation reaction.

**A distant observer sees:**
- The particle orbits the earth.
- Radiation
- Energy loss
- The particle spirals inward, not on a geodesic.

**A local observer sees:**
- Particle moves along a geodesic — “as straight as possible.”
- No radiation
- No energy loss
- The particle remains on the same geodesic?
Final resolution

A careful local observer: \( \mathbf{E} = \mathbf{E}^S + \mathbf{E}^R \)

- The Coulomb field distorted by tidal forces,
  \[
  \mathbf{E}^S = \frac{q}{r^2} \hat{r} + \text{distortion}
  \]
  \[
  \nabla \cdot \mathbf{E}^S = 4\pi q \delta(\vec{x}).
  \]

- A small, nearly constant electric field
  \[
  \mathbf{E}^R \approx \text{const}
  \]
  \[
  \nabla \cdot \mathbf{E}^R = 0.
  \]

There appears to be no local source for \( \mathbf{E}^R \).

The particle moves according to the Lorentz force law and spirals inward.

A careful local observer sees no radiation and no effect that he would be compelled to describe as radiation reaction.
Details — Scalar field

The actual inhomogeneous solution to the field equation:
\[ \Box^2 \psi = -4\pi q \delta(\vec{x}(t)) \]

Retarded solution:

\[ \psi_{\text{ret}}(p) = \frac{qu(p,p')}{r} \bigg|_{\tau_{\text{ret}}} - \int_{-\infty}^{\tau_{\text{ret}}} q v[p, p'(\tau)] \, d\tau \]

In flat spacetime \( u(p, p') = 1 \) and \( v(p, p') = 0 \).

The “tail” comes from the scattering of \( \psi \) off the curvature of spacetime. For free-fall, the self-force comes from just the “tail” part (based on local energy-momentum conservation arguments).

In curved spacetime \( u(p, p') \) and \( v(p, p') \) can be determined by local expansions when \( p \) and \( p' \) are close to each other.
A second inhomogeneous solution to the field equation:

\[ \Box^2 \psi = -4\pi q \delta(\vec{x}(t)) \]

Symmetric (in time) solution:

\[ \psi^S = \frac{qu}{2r} \bigg|_{\tau_{\text{ret}}} + \frac{qu}{2r} \bigg|_{\tau_{\text{adv}}} + \frac{1}{2} \int_{\tau_{\text{ret}}}^{\tau_{\text{adv}}} q \nu(p, \tau) \, d\tau \]

\( \psi^S \) can is known analytically from a local expansion, and is the distorted coulomb field that the local observer measures.

Actual field

\[ \psi_{\text{ret}} = \psi^S + \psi^R \]

\[ \Box^2 \psi_{\text{ret}} = \Box^2 \psi^S + \Box^2 \psi^R \]

\[ -4\pi q \delta(\vec{x}) = -4\pi q \delta(\vec{x}) + \Box^2 \psi^R \]

so that

\[ \Box^2 \psi^R = 0 \]
Summary

\[ \psi^{\text{ret}} = \psi^S + \psi^R \]

Actual, retarded field:

\[ \psi^{\text{ret}}(p) = \left. \frac{q u(p, p')}{r} \right|_{\tau^{\text{ret}}} - \int_{-\infty}^{\tau^{\text{ret}}} q v[p, p'(\tau)] \, d\tau \]

Symmetric (in time) field:

\[ \psi^S = \left. \frac{q u}{2r} \right|_{\tau^{\text{ret}}} + \left. \frac{q u}{2r} \right|_{\tau^{\text{adv}}} + \frac{1}{2} \int_{\tau^{\text{ret}}}^{\tau^{\text{adv}}} q v(p, \tau) \, d\tau \]

Homogeneous radiation reaction field:

\[ \psi^R = \psi^{\text{ret}} - \psi^S \]

\[ \psi^R = \left. \frac{q u}{2r} \right|_{\tau^{\text{ret}}} - \left. \frac{q u}{2r} \right|_{\tau^{\text{adv}}} - \int_{\infty}^{\tau^{\text{ret}}} q v(p, \tau) \, d\tau + \frac{1}{2} \int_{\tau^{\text{ret}}}^{\tau^{\text{adv}}} q v(p, \tau) \, d\tau \]
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\( \ddot{a} \sim m/L^2 \)

and \( \ddot{a} \) is small if \( m \ll L \)
A small mass $m$ orbits a large black hole and slowly emits gravitational waves. A local observer measures the gravitational field:

- The $m/r$ field distorted by tidal forces, $h^S_{ab}$.
- An external field consisting of the background combined with a small perturbing field, $g_{ab} + h^R_{ab}$.
- The local observer cannot distinguish $h^R_{ab}$ from $g_{ab}$. Together they satisfy the Einstein equations through first order in $m$.
- The mass $m$ moves along a geodesic of this external gravitational field.

With local measurements, the observer sees:

- No radiation,
- No local source for the “external field,”
- No effect that he would be compelled to describe as radiation reaction.

The observer sees only a small mass coasting along a geodesic of the “external” gravitational field with acceleration of order $m^2$. 