Radiation reaction and the self-force in extreme mass ratio binary systems

Steve Detweiler – University of Florida
collaborators:
Luz Maria Diaz-Rivera, Dong-Hoon Kim,
Eirini Messaritaki, Eric Poisson and Bernard Whiting

In the past few years a number of technical hurdles have been overcome in our understanding of radiation reaction in curved spacetime. We now have definitive methods to regularize the gravitational self-force and to remove the confusion of gauge issues in actual calculations.

In the next few years self-force calculations will be applied to extreme mass ratio binary systems. Our group will focus on the isco of the Schwarzschild geometry, tests of the extensions of post-Newtonian approximation into strong-field regimes, perturbative 2nd order gravitational wave forms, and the time dependence of the Carter constant for orbits in the Kerr geometry.

Radiation Reaction and the Equivalence Principle

- Brief discussion of the Equivalence Principle
- Classical descriptions of radiation reaction.
- An apparent paradox involving radiation reaction and the principle of equivalence.
- The resolution of the paradox.
- Some details.

Einstein's Equivalence Principle

Uniform gravitational field $=$ uniformly accelerating reference frame

Uniform gravitational field $-$ uniformly accelerating reference frame

$=$ no gravitational field

$\Rightarrow$ A small object at rest remains at rest.

$\Rightarrow$ A small object moves along a geodesic of spacetime,

requires that $L_{\text{obj}} \ll \mathscr{R}$, where $\mathscr{R}$ is a length scale of the background.

In relativity, acceleration measures the deviation from geodesic motion.

The laws of physics should be the same in every freely falling "laboratory."

In elementary electricity and magnetism there are two common types of problems:

Given q's and i's, find $\mathbf{E}$'s and $\mathbf{B}$'s (field eqns)

$$\nabla \cdot \mathbf{E} = 4\pi\rho \qquad \nabla \times \mathbf{B} = 4\pi\mathbf{J} + \partial\mathbf{E}/\partial t$$

Given $\mathbf{E}$'s and $\mathbf{B}$'s find the motion of q's and i's. (dynamical eqns)

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

It is implicit that a point charge does not interact with its own "singular field."

What is the charged particle's own "singular field"?
Ans: The coulomb field, for a charge at rest in flat spacetime.

First focus on radiation reaction:

- What does Jackson say about radiation reaction?

$$\mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{rad}} = m\dot{\mathbf{v}} \qquad \text{Abraham-Lorentz force}$$

$$\mathbf{F}_{\text{rad}} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\mathbf{v}} = \frac{2}{3}\frac{e^2}{mc^3}\dot{\mathbf{F}}_{\text{ext}} \qquad \text{calculates } \mathbf{F}_{\text{rad}}$$

- What did Dirac say about radiation reaction?

$$\mathbf{E}_{\text{ret}} = \tfrac{1}{2}(\mathbf{E}_{\text{ret}} + \mathbf{E}_{\text{adv}}) + \tfrac{1}{2}(\mathbf{E}_{\text{ret}} - \mathbf{E}_{\text{adv}}) \qquad \text{defines } \mathbf{E}_{\text{rad}}$$

$$\mathbf{E}_{\text{ret}} = \tfrac{1}{2}(\mathbf{E}_{\text{ret}} + \mathbf{E}_{\text{adv}}) + \mathbf{E}_{\text{rad}} \qquad \text{explains } \mathbf{F}_{\text{rad}}$$

$\tfrac{1}{2}(\mathbf{E}_{\text{ret}} + \mathbf{E}_{\text{adv}})$	$\mathbf{E}_{\text{rad}}$
Coulomb	radiation
inhomogeneous	homogeneous
singular	smooth

Dirac calculated the flux of energy and momentum across a sphere enclosing the charge and deduced that radiation reaction arises from the interaction of the charge with $\mathbf{E}_{\text{rad}}$.

The equivalence principle creates an apparent paradox:

Does a charged particle, falling around the Earth, radiate?

Yes:

The charged particle accelerates.

Maxwell's equations $\Rightarrow$ radiation.

No:

A particle in free-fall is not accelerating from a local point of view.

A local observer must see no radiation.

A simple resolution of the paradox

Both descriptions are correct.

- Radiation is a global phenomenon not a local one.
- A distant observer sees radiation.
- A local observer sees no radiation.
- Radiation comes from the distortion of $\mathbf{E}$'s and $\mathbf{B}$'s by the curvature of spacetime.

The paradox reappears when we consider radiation reaction.

A distant observer sees:

- The charged particle orbits the earth
- Radiation
- Energy loss
- The particle spirals inward, not on a geodesic

A local observer sees:

- Particle moves along a geodesic — “as straight as possible.”
- No radiation
- No energy loss
- The particle remains on the same geodesic?

Final resolution

A *careful* local observer measures the field
and observes two parts: $\mathbf{E} = \mathbf{E}^{\mathsf{S}} + \mathbf{E}^{\mathsf{R}}$

- The Coulomb field distorted by gravitational tidal forces:

$$\mathbf{E}^{\mathsf{S}} = \frac{q}{r^2}\hat{\mathbf{r}} + \text{ distortion}$$

$$\nabla \cdot \mathbf{E}^{\mathsf{S}} = 4\pi q \delta(\vec{x})$$

- A small, nearly constant electric field

$$\mathbf{E}^{\mathsf{R}} \approx \text{const}$$

$$\nabla \cdot \mathbf{E}^{\mathsf{R}} = 0.$$

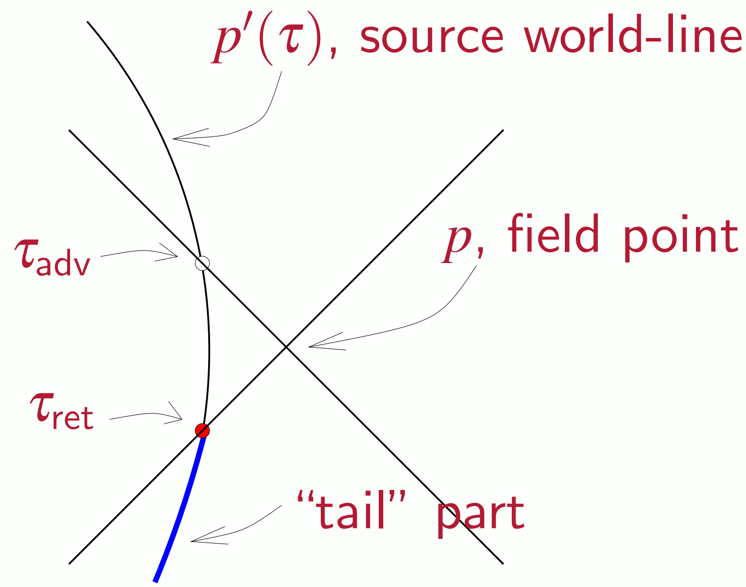
There appears to be no local source for $\mathbf{E}^{\mathsf{R}}$.
The Lorentz force law has the particle spiral inward.

A careful local observer sees no radiation and no effect
that he would be compelled to describe as radiation reaction.

Details — Scalar field

The actual inhomogeneous solution to the field equation via Green's functions

$$\Box^2 \psi = -4\pi q \, \delta(\vec{x}(t))$$



Retarded solution:

"direct" "tail"

$$\psi^{\text{ret}}(p) = \left.\frac{q u(p,p')}{r}\right|_{\tau_{\text{ret}}} - \int_{-\infty}^{\tau_{\text{ret}}} q \, v[p, p'(\tau)] \, d\tau$$

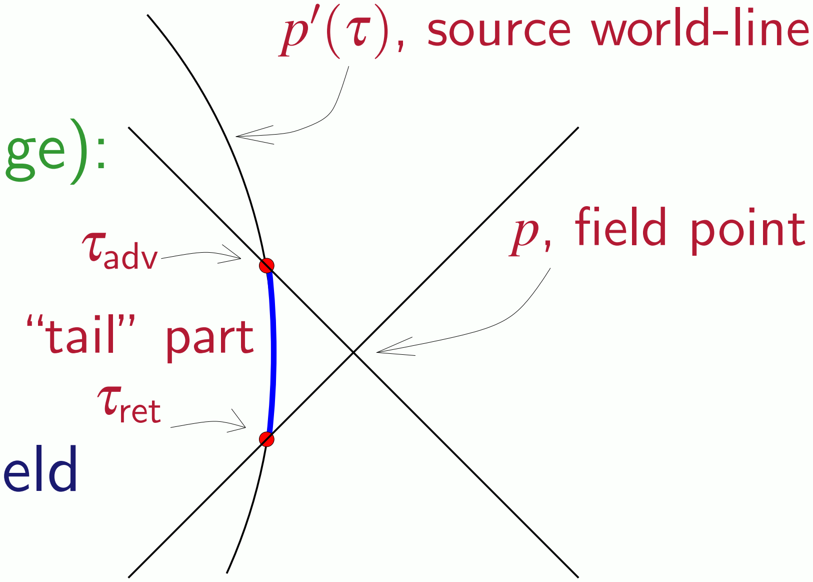
In flat spacetime $u(p,p') = 1$ and $v(p,p') = 0$.

The "tail" comes from the scattering of ψ off the curvature of spacetime.
For free-fall, the self-force comes from just the "tail" part
based on local energy-momentum conservation arguments. DeWitt and Brehme

In curved spacetime $u(p,p')$ and $v(p,p')$ can be determined by local expansions when p and p' are close to each other. Hadamard expansion

A new Green's function for:

$$\Box^2\psi = -4\pi q\delta(\vec{x}(t))$$



Singular and Symmetric (under interchange):

$$\psi^{\text{S}} = \frac{qu}{2r}\Big|_{\tau_{\text{ret}}} + \frac{qu}{2r}\Big|_{\tau_{\text{adv}}} + \frac{1}{2}\int_{\tau_{\text{ret}}}^{\tau_{\text{adv}}} q\,v(p,\tau)\,d\tau$$

ψ^{S} is known analytically from a local expansion and is the distorted coulomb field that the local observer measures.

Ψ^{S} exerts no force on its own particle.

ψ^{S} and ψ^{ret} satisfy the same field equation with the same source.

Actual field: $\psi^{\text{ret}} = \psi^{\text{S}} + \psi^{\text{R}}$

$$\begin{aligned} \Box^2\psi^{\text{ret}} &= \Box^2\psi^{\text{S}} + \Box^2\psi^{\text{R}} \\ -4\pi q\,\delta(\vec{x}) &= -4\pi q\,\delta(\vec{x}) + \Box^2\psi^{\text{R}} \end{aligned}$$

so that $\qquad \Box^2\psi^{\text{R}} = 0$

Summary

$$\psi^{\text{ret}} = \psi^{\text{S}} + \psi^{\text{R}}$$

Actual, retarded field:

$$\psi^{\text{ret}}(p) = \left.\frac{qu(p,p')}{r}\right|_{\tau_{\text{ret}}} - \int_{-\infty}^{\tau_{\text{ret}}} qv[p,p'(\tau)]\,d\tau$$

Singular Symmetric-under-interchange Source S-field:

$$\psi^{\text{S}} = \left.\frac{qu}{2r}\right|_{\tau_{\text{ret}}} + \left.\frac{qu}{2r}\right|_{\tau_{\text{adv}}} + \frac{1}{2}\int_{\tau_{\text{ret}}}^{\tau_{\text{adv}}} qv(p,\tau)\,d\tau$$

Regular Radiation-Reaction Remainder R-field:

$$\psi^{\text{R}} = \psi^{\text{ret}} - \psi^{\text{S}}$$

Radiation reaction and gravitational waves

A small mass μ orbits a large black hole and slowly emits gravitational waves.
A local observer measures the gravitational field and distinguishes two parts:

1. The μ/r field distorted by tidal forces, h^{S}_{ab}.
2. An external field consisting of the background combined with a small perturbing field, $g_{ab} + h^{\mathsf{R}}_{ab}$.

- The local observer cannot distinguish h^{R}_{ab} from g_{ab}. Together they satisfy the Einstein equations through first order in μ.
- The mass μ moves along a geodesic of this external gravitational field.

With local measurements, the observer sees:

- No radiation,
- No local source for the "external field,"
- No effect that he would be compelled to describe as radiation reaction.

The observer sees only a small mass coasting along a geodesic of the "external" gravitational field with acceleration of order μ^2, or smaller.

The laws of physics are the same in every freely falling laboratory.

Goals for our group for the next few years:

- Self-force effect on the ISCO of Schwarzschild: In or out?
- Test the validity of post-Newtonian results, when pushed into a strong field regime, via comparison with self-force calculations.
- Gravitational waveforms for LIGO and LISA:
  – First order perturbation analysis fails at $O(M/\mu)$ orbits when tracking the phase for quasi-stationary inspiral.
  – Second order perturbation analysis would increase the number of orbits by a factor of $O(\sqrt{M/\mu})$.

System: Small mass μ orbiting a much larger, Schwarzschild black hole M.

- Past: Newtonian order gravitational self-force effects
- Present: Scalar-field self-force calculations

  PRD70, 124018(2004), gr-qc/0410011
- Future: Gravitational self-force calculations

PAST:

In General Relativity a particle of infinitesimal mass moves along a geodesic, say a circular orbit centered on a black hole at radius r.

A particle of small but finite mass μ does *not* orbit the center of the black hole. Rather, it orbits the center of mass of the system—this is a Newtonian order self-force effect:

$$\Omega^2 = \frac{M}{r^3(1+\mu/M)^2} \approx \frac{M}{r^3}(1-2\mu/M)$$

$$\text{or} \qquad \frac{\Delta\Omega}{\Omega} = -\frac{\mu}{M}.$$

The finite μ influences the motion of M, which determines the gravitational field within which μ moves. This back action of μ upon its own motion is the hallmark of a self force.

The dependence of Ω on μ, via the "reduced mass" formalism, is properly described as a Newtonian-order self-force effect.

PRESENT:

How we do a self-force calculation:

1. Pick a geodesic for μ.
2. Use Regge-Wheeler to find h^{ret}_{ab}. *Numerical*
3. h^{ret}_{ab} is singular at the particle—it must be regularized:
   (a) Calculate the **S**ingular, **S**ource field $h^{\mathsf{S}}_{ab} \approx$ "μ/r" in the neighborhood of μ, with locally inertial coordinates. *Analytical*
   (b) Subtract $h^{\mathsf{ret}}_{ab} - h^{\mathsf{S}}_{ab} \equiv h^{\mathsf{R}}_{ab}$, to obtain the **R**egular **R**emainder.
   (c) h^{R}_{ab} is guaranteed to be a solution of the vacuum, perturbed Einstein equations in a neighborhood of μ.
4. The Bianchi identity implies that μ moves along a geodesic of $g_{ab} + h^{\mathsf{R}}_{ab}$. This appears as acceleration in the Schwarzschild geometry:

$$u^a \underset{(g)}{\nabla}_a u^b = u^a u^c (g^{bd} + u^b u^d)(\underset{(g)}{\nabla}_a h^{\mathsf{R}}_{cd} - \frac{1}{2} \underset{(g)}{\nabla}_d h^{\mathsf{R}}_{ac}).$$

and the RHS is the "gravitational self force/μ."

5. The self force includes the effects of "radiation reaction."

Recent results

PRD70, 124018(2004)

Scalar field self-force effects:

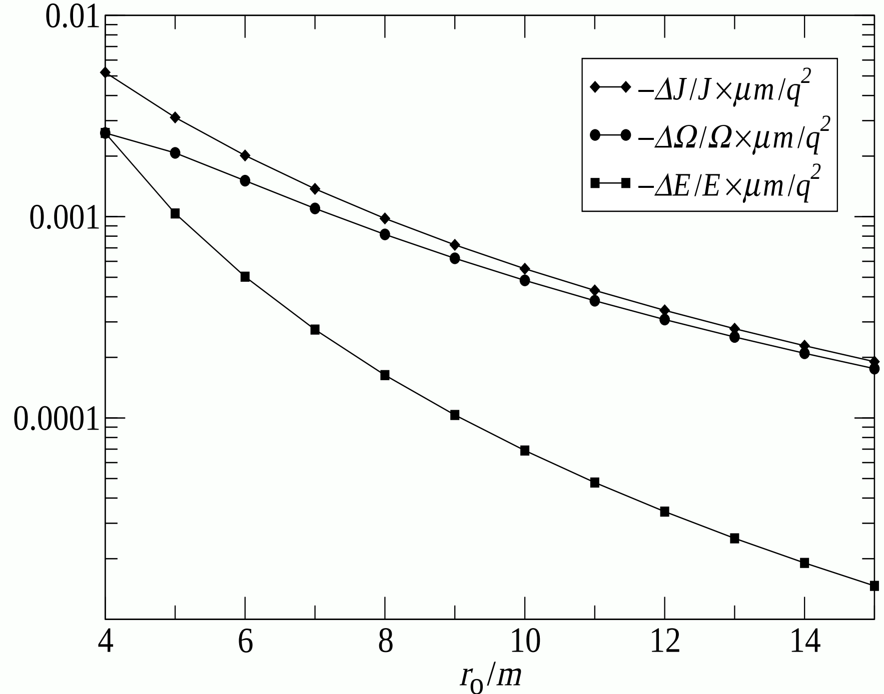
1. Circular orbits:

   The self force is outward

   $\Delta\Omega/\Omega(r) < 0$

   $\Delta E/E(r) < 0$

   $\Delta J/J(r) < 0$



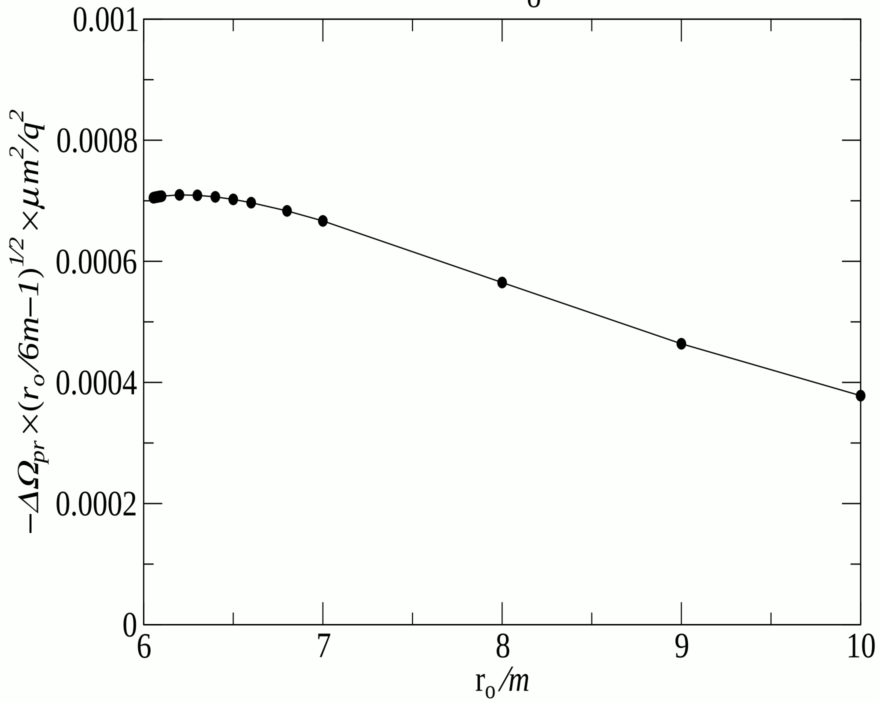
2. Slightly eccentric orbits:

   $\Omega_r^2 = \frac{M}{r^4}(r - 6M)$

   $\Omega_{\text{prec}} = \Omega - \Omega_r$

   $\Delta\Omega_r/\Omega_r(r) > 0$

   $\Delta\Omega_{\text{prec}}/\Omega_{\text{prec}}(r) < 0$



Scalar field self-force effect on the ISCO: *PRD70, 124018(2004)*

With $\Omega_r^2 = \frac{M}{r^4}(r-6M)$, for $r > 6M$, $\Omega_r^2 > 0$, Ω_r is real, and radial oscillations of the orbit are stable. For $r < 6M$, $\Omega_r^2 < 0$, Ω_r is imaginary, a small radial perturbation grows exponentially and the orbit is unstable.

The ISCO is defined as that circular orbit for which $\Omega_r = 0$.

$$r_{\mathsf{ISCO}} = 6M \qquad \Omega_{\mathsf{ISCO}} = \sqrt{\frac{M}{(6M)^3}} = 0.068M^{-1}$$

The effect of the scalar-field self-force moves the radius of the innermost stable circular orbit inward:

$$\Delta r_{\mathsf{ISCO}} = -0.122701 \times q^2/\mu .$$

Ω_{ISCO} increases:

$$\frac{\Delta\Omega_{\mathsf{ISCO}}}{\Omega_{\mathsf{ISCO}}} = 0.0291657 \times q^2/\mu M$$

Before considering gravitational self-force problems we must face

Gauge Issues:

- When the metric on a manifold is perturbed, the coordinates in use become inherently ambiguous at $O(h_{ab}) \sim O(\mu/M)$.
- Resolving the ambiguity in some manner results in a "gauge choice."
- This ambiguity makes it more difficult than usual to distinguish physical effects from coordinate effects.
- The "gauge" ambiguity allows $h_{ab} \to h_{ab} - 2\nabla_{(a}\xi_{b)}$ without a change in the physical situation; ξ^a is arbitrary and is the "gauge" vector.
- The gravitational self force depends upon the gauge choice.
- A gauge choice always exists such that the self force is zero for a time that is much longer than the dynamical timescale.
- The coordinate position (radius) of μ is *not* gauge invariant. It depends upon the gauge choice.
- For self-force calculations it appears best to focus on quantities which are gauge invariant. These are usually associated with clean, physical observations.
- A physical observation made at infinity usually gives a gauge invariant quantity.

Future:

Gauge invariant quantities for circular or slightly eccentric orbits include Ω, Ω_r and Ω_{prec} — these quantities could be measured by an observer at infinity.

- $\Delta\Omega$, $\Delta\Omega_r$, $\Delta\Omega_{\mathrm{prec}}$.

- Comparison of these results with post-Newtonian analyses, pushed to the strong field regime.

- Gravitational self-force effects on the ISCO:
  $\Delta\Omega_{\mathrm{ISCO}}/\Omega_{\mathrm{ISCO}}$
  $\Delta R_{\mathrm{ISCO}}/R_{\mathrm{ISCO}}$ **No! R is not gauge invariant.**
  $\Delta E_{\mathrm{ISCO}}/E_{\mathrm{ISCO}}$ **No! E requires a second order calculation.**

The $O(\mu)$ effects on $E \equiv -t^a u_a$ and $J \equiv \phi^a u_a$ are not well defined, not gauge invariant and cannot be calculated *at this level of approximation*. t^a and ϕ^a are not Killing vectors of the perturbed geometry.

Self-force effects on phase with quasi-stationary inspiral in Schwarzschild.

The $O(\mu)$ effects on $E \equiv -t^a u_a$ and $J \equiv \phi^a u_a$ are not well defined, not gauge invariant and cannot be calculated *at this level of approximation*. t^a and ϕ^a are not Killing vectors of the perturbed geometry.

For the purpose of this discussion, ignore energy flux down the hole, and assume that the system is relativistic. t is retarded time;

M_∞ is the total mass as measured at infinity and is gauge invariant.

Define E (in a gauge invariant manner) from $M_\infty \equiv M + \mu E$.

Let $E = E_{\text{geod}}[1 + \Delta_{\text{2nd}} + \cdots]$,
where $E_{\text{geod}} = -t^a u_a$,
and $E_{\text{geod}} \times \Delta_{\text{2nd}}$ is the $E_{\text{geod}} \times O(\mu/M)$ contribution
to E from 2nd order perturbation theory.

Let dE/dt be the energy loss through gravitational radiation, with
$dE/dt = dE/dt_{\text{1st}}[1 + \Delta_{\text{2nd}} + \cdots]$.

Here, dE/dt_{1st} is evaluated by using the usual Regge-Wheeler style analysis, as was done in the 70's, and $dE/dt_{\text{1st}} \times \Delta_{\text{2nd}}$ is the $dE/dt_{\text{1st}} \times O(\mu/M)$ contribution to dE/dt from 2nd order perturbation theory.

Let Ω_o be the frequency at $t = 0$. For quasi-circular inspiral the phase is determined by

$$\begin{aligned} \phi &= \int_0^t \Omega(E(t))dt \\ &= \int_0^t \left(\Omega_o + t\left[\frac{d\Omega}{dE}\frac{dE}{dt}\right]_{\text{1st}} [1+\Delta_{\text{2nd}}+\cdots]\right)dt \end{aligned} \tag{1}$$

After integration, consider the size of the contribution to the phase of each term:

$$\frac{1}{2}t^2\left[\frac{d\Omega}{dE}\frac{dE}{dt}\right]_{\text{1st}}[1+\Delta_{\text{2nd}}+\cdots] \approx \frac{1}{2}t^2\left[O\left(\frac{\mu}{M^3}\right)\right]_{\text{1st}}\left(1+\left[O\left(\frac{\mu}{M}\right)\right]_{\text{2nd}}+\cdots\right)$$

• If radiation reaction is ignored, then the $\frac{1}{2}t^2 \times O(\mu/M^3)$ term is not included, which leads to a phase error of one full cycle after a time $t_{\text{1st}} = O\left(M\sqrt{\frac{4\pi M}{\mu}}\right)$.

• If only first-order radiation reaction is included, then the $\frac{1}{2}t^2 \times O(\mu^2/M^4)$ terms from second order effects create a phase error of one full cycle after a time $t_{\text{2nd}} = O\left(\frac{M^2}{\mu}\sqrt{4\pi}\right)$.

After integration, consider the size of the contribution to the phase of each term:

$$\frac{1}{2}t^2\left[\frac{d\Omega}{dE}\frac{dE}{dt}\right]_{1\text{st}}[1+\Delta_{2\text{nd}}+\cdots]\approx\frac{1}{2}t^2\left[O\left(\frac{\mu}{M^3}\right)\right]_{1\text{st}}\left(1+\left[O\left(\frac{\mu}{M}\right)\right]_{2\text{nd}}+\cdots\right)$$

• If radiation reaction is ignored, then the $\frac{1}{2}t^2 \times O(\mu/M^3)$ term is not included, which leads to a phase error of one full cycle after a time $t_{1\text{st}} = O\left(M\sqrt{\frac{4\pi M}{\mu}}\right)$.

• If only first-order radiation reaction is included, then the $\frac{1}{2}t^2 \times O(\mu^2/M^4)$ terms from second order effects create a phase error of one full cycle after a time $t_{2\text{nd}} = O\left(\frac{M^2}{\mu}\sqrt{4\pi}\right)$.

• If second-order radiation reaction is included, then the $\cdots$ terms are $\frac{1}{2}t^2 \times O(\mu^3/M^5)$ and create a phase error of one full cycle after a time $t_{\dots} = O\left(\frac{M^2}{\mu}\sqrt{\frac{4\pi M}{\mu}}\right)$.

• Geodesic motion fails to track the phase after $O\left(\sqrt{\frac{4\pi M}{\mu}}\right)$ orbits

• First order perturbation theory fails to track the phase after $O\left(\frac{M}{\mu}\sqrt{4\pi}\right)$ orbits.

• Second order perturbation theory fails to track the phase after $O\left(\frac{M}{\mu}\sqrt{\frac{4\pi M}{\mu}}\right)$ orbits.

Great Expectations:

- Within the next few years we will be able to do 2nd order perturbation theory for a small mass orbiting a Schwarzschild black hole.

- Orbits of the Kerr geometry will continue to present a challenge.

- The Carter constant C is not associated with any "gauge invariant" quantity, the way that E is. So 1st order perturbation theory is not easily applied to dC/dt via a flux integral.

- 1st order perturbation theory, coupled with self-force analysis will provide 1st order wave forms for Kerr (recent work by Mino), if we can determine the metric perturbations.

- Metric perturbations of Kerr are accessible in principle from the Teukolsky analysis of ψ_0 and ψ_4. We should be able to do this, but it has not yet been attempted.

- Metric perturbations of Kerr might also be determined via a direct, non-separable, analysis which is not based upon the Teukolsky equation.