Perspective on gravitational self-force analyses

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Abstract. A point particle of mass μ moving on a geodesic creates a perturbation h_{ab}, of the spacetime metric g_{ab}, that diverges at the particle. Simple expressions are given for the singular μ/r part of h_{ab} and its tidal distortion caused by the spacetime. This singular part h^{S}_{ab} is described in different coordinate systems and in different gauges. Subtracting h^{S}_{ab} from h_{ab} leaves a regular remainder h^{R}_{ab}. The self-force on the particle from its own gravitational field adjusts the world line at $\mathrm{O}(\mu)$ to be a geodesic of $g_{ab} + h^{\mathrm{R}}_{ab}$; this adjustment includes all of the effects of radiation reaction. For the case that the particle is a small non-rotating black hole, we give a uniformly valid approximation to a solution of the Einstein equations, with a remainder of $\mathrm{O}(\mu^2)$ as $\mu \to 0$.

An example presents the actual steps involved in a self-force calculation. Gauge freedom introduces ambiguity in perturbation analysis. However, physically interesting problems avoid this ambiguity.

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1. Introduction

A description of motion always entails approximations and abstractions. The motion of a small black hole through spacetime is clearly not a geodesic of the actual, physical spacetime geometry. After all, the "center" of a black hole is inside the event horizon, where the geometry is unknown. Nevertheless, if the mass of the hole is sufficiently small in comparison with a length scale of spacetime, then the motion is approximately geodesic on an abstract spacetime which is described as "spacetime with the gravitational field of the black hole removed". Much of this manuscript focuses upon the meaning of this last phrase.

In general relativity, an object of infinitesimal mass and size moves through a background spacetime along a geodesic. If the particle has a small but finite mass μ then its world line Γ deviates from a geodesic of the background by an amount proportional to μ. This deviation is sometimes described as resulting from the "self-force" of the particle's own gravitational field acting upon itself and includes the effects which are often referred to as radiation reaction.

In the literature the phrase "gravitational self-force" often refers to precisely the right hand side of (11), given below. As emphasized by Barack and Ori [1], the value of this quantity depends upon the gauge being used (see section 9.3) and is, thus, ambiguous. In this manuscript the phrase "gravitational self-force" is only used in an imprecise, generic way to describe any of the effects upon an object's motion which are proportional to its own mass.

1.1. Newtonian self-force example

Newtonian gravity presents an elementary example of a self-force effect [2]. A small mass μ in a circular orbit of radius R about a more massive companion M has an angular frequency Ω given by

$$\Omega^2 = \frac{M}{R^3(1+\mu/M)^2}. \tag{1}$$

When μ is infinitesimal, the large mass M does not move, the radius of the orbit R is equal to the separation between the masses and $\Omega^2 = M/R^3$. However when μ is finite but still small, both masses orbit their common center of mass with a separation of $R(1+\mu/M)$, and the angular frequency is as given in (1). The finite μ influences the motion of M, which influences the gravitational field within which μ moves. This back action of μ upon its own motion is the hallmark of a self-force, and the μ dependence of (1) is properly described as a Newtonian self-force effect. When μ is much less than M, an expansion of (1) provides

$$\Omega^2 \approx \frac{M}{R^3}\left[1 - 2\mu/M + \mathrm{O}(\mu^2/M^2)\right]. \tag{2}$$

The finite mass ratio μ/M changes the orbital frequency by a fractional amount

$$\frac{\Delta\Omega}{\Omega} = -\frac{\mu}{M}. \tag{3}$$

In this manuscript we describe any such $\mathrm{O}(\mu/M)$ effect on the motion as being a "gravitational self-force" effect. Below, we see that the self-force effects for gravity include all of the consequences of what is often referred to as "radiation reaction." However, we also see that a local observer, near μ deep inside the wave-zone and not privy to global spacetime information, is unable to distinguish radiation reaction and the other parts of the gravitational self-force from pure geodesic motion, at this level of approximation.

1.2. Electromagnetic radiation reaction in flat spacetime

The Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{4}$$

describes the interaction of a point charge q with an electromagnetic field. In an elementary electricity and magnetism course, it is implicit that q's own electromagnetic field is not to be included on the right hand side—after all for a point charge $\mathbf{E}$ is infinite at the very location where it is to be evaluated in (4). Thus, the electromagnetic field of (4) is an "external" field, whose source might be, say, the parallel plates of a capacitor but does not include the charge q itself.

Abraham and Lorentz first derived the radiation reaction force on a point charge [3]

$$\mathbf{F}_{\mathrm{rad}} = \frac{2}{3}\frac{q^2}{c^3}\ddot{\mathbf{v}} \tag{5}$$

in terms of the changing acceleration of q. This equation may be interpreted in a perturbative sense: Let q have a small mass and be oscillating on the end of a spring.

At lowest order in the perturbation, q executes simple harmonic motion. At first order in the perturbation, the right hand side of (5) is evaluated by use of a $\ddot{\mathbf{v}}$ consistent with the harmonic motion. The resulting $\mathbf{F}$ is a small damping force which removes energy from the system at just the proper rate to account for the outward energy flux of radiation.

A great value of (5) resides in its elementary use by a theorist to calculate the radiation reaction force.

A drawback of (5) is the apparent obscuration of the root cause of this force. Charges interact with electromagnetic fields via (4). Yet, no electromagnetic field is present in (5). Imagine a local observer extremely close to q, deep within the wave zone, and with a length scale very much smaller than that associated with the oscillations. This observer correctly interprets the majority of the acceleration of q as resulting from the coupling to the spring. The local observer is unaware of the radiation—a non-local concept; yet, he must explain the deviation from pure harmonic motion resulting from $\mathbf{F}_{\mathrm{rad}}$ as a consequence of the interaction of q with some external field via (4). The Abraham-Lorentz analysis correctly *calculates* the electromagnetic self-force. But it does not *explain* this self-force in terms of the charge interacting with an external electromagnetic field.

Dirac [4] removes this drawback by providing an interpretation of (5) as a direct consequence of (4), with the electromagnetic field on the right hand side being an external field of indeterminate origin to the local observer. Dirac uses the conservation of the electromagnetic stress-energy tensor in a world-tube surrounding q, and ultimately takes the limit of vanishing radius of the world-tube. One consequence of his analysis is that the half-advanced plus half-retarded field $F^{\mathrm{S}}_{ab} = \frac{1}{2}(F^{\mathrm{ret}}_{ab} + F^{\mathrm{adv}}_{ab})$ of q exerts no force on q itself, even though the field is formally singular in the point charge limit. We call the actual field F^{act}_{ab}, and the remainder $F^{\mathrm{R}}_{ab} = F^{\mathrm{act}}_{ab} - F^{\mathrm{S}}_{ab}$ is a vacuum solution of Maxwell's equations. F^{R}_{ab} substituted into the right hand side of (4) yields (5), as shown by Dirac.

A local observer measures the electromagnetic field in the vicinity of q, but with no information regarding boundary conditions or distant radiation, he can make no conclusions as to the detailed cause or source of the field. However, in the perturbative sense described above, the observer can calculate the singular field F^{S}_{ab} in the vicinity of q. He can subtract this singular field F^{S}_{ab} from the actual, measured field F^{act}_{ab} to obtain

$$F^{\mathrm{R}}_{ab} = F^{\mathrm{act}}_{ab} - F^{\mathrm{S}}_{ab}. \tag{6}$$

The charge q then interacts with the resulting regular source-free electromagnetic field F^{R}_{ab} via (4) with a resulting small perturbation in its motion. Thus, a local observer naturally explains the damping of the harmonic motion as a consequence of q interacting with an external, locally source-free field F^{R}_{ab}. However, with no global information regarding boundary conditions he would not be able to determine the cause or source of this external field. In particular the local observer would see no phenomenon which he would be compelled to describe as radiation reaction.

1.3. Electromagnetic radiation reaction in curved spacetime

DeWitt and Brehme's [5] pioneering analysis of electromagnetic radiation reaction in curved spacetime follows Dirac's approach and also uses the conservation of energy in a world-tube to determine the force on a point charge. Their results reduce to

Dirac's in the flat spacetime limit. However, DeWitt and Brehme find that generally $\frac{1}{2}(F_{ab}^{\rm ret} + F_{ab}^{\rm adv})$ does, in fact, exert a force on the charge in curved spacetime. After its removal from the actual field, the remainder does not serve as the electromagnetic field on the right hand side of (4) for calculating a radiation reaction force.

To simplify the remainder of this introduction we, henceforth, assume that the charge is in free fall in curved spacetime—the charge would move along a geodesic except for interaction with its own electromagnetic field; there are no springs attached.

DeWitt and Brehme use the Lorenz gauge, $\nabla_a A^a = 0$, and a Hadamard expansion to break the Green's function into the "direct" and "tail" parts with the vector potential

$$A_a^{\rm ret} \equiv A_a^{\rm dir} + A_a^{\rm tail}. \qquad (7)$$

The direct part of the retarded Green's function has support only on the past null cone, and the tail part has support only inside the past null cone. They find that the electromagnetic self-force can be described as a consequence of the particle interacting just with $A_a^{\rm tail}$,

$$F^a_{\rm rad} = q g^{ac}(\nabla_c A_b^{\rm tail} - \nabla_b A_c^{\rm tail}) u^b. \qquad (8)$$

This expression, like (5), has the great value that it can be used to calculate an electromagnetic self-force, but it shares the drawback that it does not explain the self-force in terms of a locally measurable, source free solution of the Maxwell equations. In fact $A_a^{\rm tail}$ is not in any sense a solution of the electromagnetic field equation

$$\nabla^2 A^a - R^a{}_b A^b = -4\pi J^a. \qquad (9)$$

The details of the Hadamard expansion reveal that if $A^a_{\rm tail}$ were inserted into the left hand side here, it would yield a phantom $J^a_{\rm tail}$, throughout a neighborhood of the charge. There would be no other evidence for the existence of this $J^a_{\rm tail}$. Further, if $(R_{ab} - \frac{1}{6} g_{ab} R) u^b \neq 0$, then $A_a^{\rm tail}$ is not differentiable at the particle and some version of averaging around the charge is required to compute the self-force. $A^a_{\rm tail}$ is a valuable mathematical construct which may be used to calculate the self-force from (8), but it is not associated with an actual electromagnetic field. We conclude that the DeWitt-Brehme construction correctly *calculates* the electromagnetic self-force. But it does not *explain* the self-force in terms of the charge interacting with an external electromagnetic field.

A modification [6] of the DeWitt and Brehme analysis has rectified this shortcoming. The actual vector potential may be decomposed as

$$A_a^{\rm act} \equiv A_a^{\rm S} + A_a^{\rm R}, \qquad (10)$$

where $A_a^{\rm S}$ and $A_a^{\rm R}$ are, in fact, solutions of Maxwell's equations in a neighborhood of q: $A_a^{\rm S}$ has only the charge q as its source, while $A_a^{\rm R}$ is a vacuum solution. Further, (8) yields the same force whether $A_a^{\rm R}$ or $A_a^{\rm tail}$ is inserted on the right hand side, after the possible lack of differentiability of $A_a^{\rm tail}$ is handled properly.

One nuance of the decomposition into S- and R-fields, is that the Green's function for the S-field has support at the advanced and retarded times, just as in the flat-spacetime example, above. But it also has support at the events between the retarded and advanced times—these have a spacelike separation with the field point.

The "S" and "R" decomposition provides a local observer in curved spacetime with the ability to measure the actual electromagnetic field $F_{ab}^{\rm act}$ in a neighborhood of q. He can make no conclusions as to the detailed cause or source of the field. However, in the perturbative sense described above, the observer can calculate $F_{ab}^{\rm S}$

in a neighborhood of q based upon its approximate geodesic motion. He can then subtract this singular field F^{S}_{ab} from the actual, measured field F^{act}_{ab}. The charge q then interacts with the resulting regular source-free electromagnetic field F^{R}_{ab} via (4) or (8) with a resulting small perturbation of its geodesic motion. Thus, a local observer naturally explains the lack of geodesic motion of a charge q as a consequence of q interacting with an external, locally source-free electromagnetic field. However, with no global information regarding boundary conditions he is not able to determine the cause or source of this external field. In particular, at this level of approximation the local observer sees no phenomenon which he would be compelled to describe as radiation reaction.

1.4. Gravitational self-force

The treatment of gravitational radiation reaction and self-force, in terms of Green's functions, are formally very similar to that just described for the electromagnetic field.

In some circumstances the gravitational field may be considered to have an effective stress-energy tensor consisting of terms which are quadratic in the derivatives of the metric. Mino, Sasaki and Tanaka [7] follow the DeWitt-Brehme [5] approach, but with this gravitational stress energy tensor. Ultimately, they conclude that the motion of a point mass μ satisfies

$$\mu u^b \nabla_b u^a = -\mu (g^{ab} + u^a u^b) u^c u^d (\nabla_c h^{\mathrm{tail}}_{db} - \frac{1}{2} \nabla_b h^{\mathrm{tail}}_{cd}). \tag{11}$$

In an independent analysis within the same paper, they treat μ as a small black hole moving in an external universe and use a general matched asymptotic expansion to arrive at the same conclusion. In this latter approach, the metric of the black hole is considered to be perturbed by the external universe through which it is moving. Simultaneously, the metric of the external universe is considered to be perturbed by the small mass μ moving through it. Others have used matched asymptotic expansions to describe the motion of a small black in an external universe [8, 9, 10, 11, 12, 13], but the connection between such results and radiation reaction appears not to have been made before reference [7].

Quinn and Wald [14] use an axiom based analysis of the gravitational self-force and also arrive at (11).

The form of equation (11) is equivalent, through first order in h^{tail}_{ab}, to the geodesic equation for the metric $g_{ab} + h^{\mathrm{tail}}_{ab}$. From one perspective then (11) is the gravitational equivalent of (8). Equation (11), like (8), has the great value that it can be used to calculate a gravitational self-force, but it shares the drawback that it does not explain the gravitational self-force in terms of geodesic motion in a locally measurable, source free solution of the Einstein equations. In fact, h^{tail}_{ab} is not in any sense a solution of the perturbed Einstein equation, given below in (13).

The details of the Hadamard expansion reveal that if h^{tail}_{ab} were inserted into the left hand side of (13), it would yield a phantom stress-energy tensor T^{tail}_{ab}, throughout a neighborhood of μ. There would be no other evidence for the existence of this T^{tail}_{ab}. Further, when $R_{acbd} u^c u^d \neq 0$, then h^{tail}_{ab} is not even differentiable at the particle; although details reveal that averaging around the particle is not required to compute the self-force with (11). h^{tail}_{ab} is a valuable mathematical construct which may be used to calculate the self-force from (11), but it is not associated with an actual gravitational field. We conclude that the Mino, Sasaki and Tanaka and the Quinn and Wald constructions correctly *calculate* the gravitational self-force. But they do not *explain* the self-force in terms of geodesic motion in an external gravitational field.

A modification [6] of the analysis involving $h^{\rm tail}_{ab}$ has rectified this shortcoming. The actual metric perturbation may be decomposed as

$$h^{\rm act}_{ab} \equiv h^{\rm S}_{ab} + h^{\rm R}_{ab}, \tag{12}$$

where $h^{\rm S}_{ab}$ and $h^{\rm R}_{ab}$ are, in fact, solutions of the perturbed Einstein equations (13) in a neighborhood of μ: $h^{\rm S}_{ab}$ has only the mass μ as its source, while $h^{\rm R}_{ab}$ is a vacuum solution. Further, (11) yields the same force whether $h^{\rm R}_{ab}$ or $h^{\rm tail}_{ab}$ is inserted on the right hand side.

Earlier [12], asymptotic matching was used to find an explicit expression for the leading terms in an expansion of $h^{\rm S}_{ab}$ in powers of the distance away from μ. Further, it was also shown that $h^{\rm R}_{ab} = h^{\rm act}_{ab} - h^{\rm S}_{ab}$ was at least C^1, with the given terms of the expansion for $h^{\rm S}_{ab}$, and that μ necessarily followed a geodesic of $g_{ab} + h^{\rm R}_{ab}$ up to terms of $\mathrm{O}(\mu^2/\mathcal{R}^2)$, where $\mathcal{R}$ is a length scale of the background geometry. However, at that time it was erroneously claimed [12] that the $h^{\rm R}_{ab}$ field was identical to $h^{\rm tail}_{ab}$ because both led to the same equation of motion—namely geodesic motion in $g_{ab} + h^{\rm R}_{ab}$. It was during a failing effort to demonstrate directly this equivalence that the important differences between the pair $h^{\rm S}_{ab}$ and $h^{\rm R}_{ab}$ and the pair $h^{\rm dir}_{ab}$ and $h^{\rm tail}_{ab}$ as possible solutions of the perturbed Einstein equations were discovered [6].

A small mass μ moves through a background geometry g_{ab} along a world line Γ. At the lowest order in a perturbative sense, Γ is a geodesic. The Newtonian example given in section 1.1 implies that Γ deviates from geodesic motion in g_{ab} by $\mathrm{O}(\mu/\mathcal{R})$—it is this deviation in which we are interested.

A local observer in curved spacetime has the ability to measure the actual metric $g^{\rm act}_{ab}$ in a neighborhood of μ. In a perturbative sense, the observer can calculate $h^{\rm S}_{ab}$ in a neighborhood of μ based upon its approximately geodesic motion. He can then subtract this singular field $h^{\rm S}_{ab}$ from the actual, measured field $g^{\rm act}_{ab}$. The mass μ will be observed to move along a geodesic of $g^{\rm act}_{ab} - h^{\rm S}_{ab} = g_{ab} + h^{\rm R}_{ab}$. Thus, a local observer sees geodesic motion of μ in the metric $g_{ab} + h^{\rm R}_{ab}$, which is a vacuum solution of the Einstein equations, up to a remainder of $\mathrm{O}(\mu^2)$ in a neighborhood of μ. With no global information regarding, say, the original background metric g_{ab}, he would be unable to make any measurement which would distinguish the separate parts g_{ab} and $h^{\rm R}_{ab}$ which together make up the metric through which μ is moving on a geodesic. At this level of approximation the local observer sees only geodesic motion and no phenomenon which he would be compelled to describe as radiation reaction.

1.5. Outline

Perturbation analysis, described in 2, is the heart of the self-force formalism. A variety of locally inertial coordinate systems are identified in 3. Some of the ensuing mathematics is simplified by use of notation, introduced in 4, which is convenient for describing vector and tensor harmonics in a spherically symmetric geometry.

Sections 5-7 describe the metric in the neighborhood of a small black hole as it moves through spacetime and provide an identification of the singular "S-part" of a particle's gravitational field, which exerts no force on the particle, itself. The remaining "R-part" of the particle's gravitational field is then seen to be responsible for the gravitational self-force in 8. The confusion caused by the gauge freedom inherent in the R-part is summarized in 9.

An example of a point mass in a circular orbit about a Schwarzschild black hole reveals, in section 10, how the difficulty of gauge dependence may be handled in carefully defined circumstances. Future prospects for gravitational self-force calculations are discussed in 11.

1.6. Conventions and notation

Conventions and notation are described here and again in context below. The indices a, b, c... are spacetime indices lowered and raised with the metric g_{ab} and its inverse; the derivative operator compatible with g_{ab} is ∇_a. The metric of flat Minkowskii space is η_{ab}. The indices i, j, k, l, p, q are always used as spatial indices and are raised and lowered with the flat three-metric f_{ij}. $\hat{n}_i$ is a unit radial vector in flat space.

Indices A, B,... are used to denote vector or tensor components which are tangent to a two-sphere in spherically symmetric geometries, especially those which are generated by "potential" functions as described in section 4. Spatial, symmetric trace-free tensors such as $\mathcal{E}_{ij}$ or $\mathcal{B}_{ijk}$ represent the external gravitational multipole moments, when the gravitational field field is expanded in a locally inertial coordinate system. The symbols $\mathcal{E}$ and $\mathcal{B}$ always refer to the even and odd parity moments, respectively. The scalars $\mathcal{E}^{(2)} = \mathcal{E}_{ij}\hat{n}^i\hat{n}^j$ and $\mathcal{B}^{(3)} = \mathcal{B}_{ijk}\hat{n}^i\hat{n}^j\hat{n}^k$, for examples, represent linear combinations of the $\ell = 2$ and $\ell = 3$ spherical harmonics, respectively, which depend only upon the angles θ and ϕ in the usual Schwarzschild coordinates, and are independent of t and r. the superscript (2) denotes the value of ℓ.

A small particle of mass μ moves along a world line Γ parameterized by the proper time s. p is an event on Γ. $\mathcal{R}$ is a representative length scale associated with a geodesic Γ of spacetime—$\mathcal{R}$ is the smallest of the radius of curvature, the scale of inhomogeneities, and the time scale for changes in curvature along Γ. We use $h^{\mathrm{S}}_{ab}$to represent the singular source field, while h^{μ}_{ab} is an approximation to h^{S}_{ab} based upon an asymptotic expansion.

2. First order perturbation analysis

Perturbation analysis provides the framework for an understanding of the self-force and radiation reaction on an object of small mass and size in general relativity. This begins with a background spacetime metric g_{ab} which is a vacuum solution of the Einstein equations $G_{ab}(g) = 0$. An object of small mass μ then disturbs the geometry by an amount $h_{ab} = \mathrm{O}(\mu)$ which is governed by the perturbed Einstein equations with the stress-energy tensor $T_{ab} = \mathrm{O}(\mu)$ of the object being the source,

$$E_{ab}(h) = -8\pi T_{ab} + \mathrm{O}(\mu^2). \tag{13}$$

Here $E_{ab}(h)$ is the linear, second order differential operator on symmetric, two-indexed tensors schematically defined by

$$E_{ab}(h) \equiv -\frac{\delta G_{ab}}{\delta g_{cd}} h_{cd}, \tag{14}$$

and G_{ab} is the Einstein tensor of g_{ab}, so that

$$\begin{aligned} 2E_{ab}(h) = {} & \nabla^2 h_{ab} + \nabla_a \nabla_b h - 2\nabla_{(a}\nabla^c h_{b)c} \\ & + 2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c\nabla^d h_{cd} - \nabla^2 h), \end{aligned} \tag{15}$$

with $h \equiv h_{ab} g^{ab}$ and ∇_a and $R_a{}^c{}_b{}^d$ being the derivative operator and Riemann tensor of g_{ab}. If h_{ab} is a solution of (13) then it follows from (14) that $g_{ab} + h_{ab}$ is an approximate solution of the Einstein equations with source T_{ab},

$$G_{ab}(g+h) = 8\pi T_{ab} + \mathrm{O}(\mu^2). \tag{16}$$

The Bianchi identity implies that

$$\nabla^a E_{ab}(h) = 0 \tag{17}$$

for any symmetric tensor h_{ab}; this is discussed in Appendix A. Thus, an integrability condition for (13) is that the stress-energy tensor T_{ab} be conserved in the background geometry g_{ab},

$$\nabla^a T_{ab} = \mathrm{O}(\mu^2). \tag{18}$$

Perturbation analysis at the second order is no more difficult formally than at the first. But the integrability condition for the second order equations is that T_{ab} be conserved not in the background geometry, but in the first order perturbed geometry. Thus, before solving the second order equations, it is necessary to change the stress-energy tensor in a way which is dependent upon the first order metric perturbations. This modification to T_{ab} is said to result from the "self-force" on the object from its own gravitational field and includes the dissipative effects of what is often referred to as "radiation reaction" as well as other nonlinear aspects of general relativity. This modification to T_{ab} is $\mathrm{O}(\mu^2)$ because T_{ab} itself is $\mathrm{O}(\mu)$.

A description of general, nth order perturbation analysis is given in Appendix B. The procedure is similar to that just outlined. The stress-energy tensor must be conserved with the metric $g^{(n-1)}_{ab}$ in order to solve the nth order perturbed Einstein equation (B.4) for $h^{(n)}_{ab}$. In an implementation, the task then alternates between solving the equations of motion for the stress-energy tensor and solving the perturbed Einstein equation for the metric perturbation. Similar alternation of focus between the equations of motion and the field equations is present in post-Newtonian analyses.

For many interesting situations the object is much smaller than the length scale of the geometry through which it moves. We expect, then, that the detailed structure of the source should be unimportant in determining its subsequent motion.

To focus on those details of the self-force which are independent of the object's structure we first attempt to model the object by an abstract point particle with no spin angular momentum or internal structure. The stress-energy tensor of a point particle is

$$T^{ab} = \mu \int_{-\infty}^{\infty} \frac{u^a u^b}{\sqrt{-g}} \delta^4(x^a - X^a(s))\,\mathrm{d}s \tag{19}$$

where $X^a(s)$ describes the world line Γ of the particle in some coordinate system as a function of the proper time s along the world line.

The naive replacement of a small object by a delta-function distribution for the stress-energy tensor is satisfactory at first order in the perturbation analysis. The integrability condition (18) requires the conservation of the perturbing stress-energy tensor. For a point particle this implies that the world line Γ of the particle is an approximate geodesic of the background metric g_{ab}, with $u^a \nabla_a u^b = \mathrm{O}(\mu)$ (*cf* Appendix C). The solution of (13) is formally straightforward, even for a distribution valued source. This procedure has been used many times to study the emission of gravitational waves by a point mass orbiting a black hole [15, 16, 17].

A difficulty appears with the second order integrability condition (B.10), with $n = 2$. This condition seems to require that the particle move along a geodesic of $g_{ab} + h_{ab}$. But h_{ab} is singular precisely at the location of the particle. To rectify this situation we look for a method to identify and to remove the singular part $h^{\rm S}_{ab}$ of the point particle's metric perturbation and, thus, to find the remaining $h^{\rm R}_{ab}$. We would then have the expectation that the point particle would move along a geodesic of the abstract, perturbed geometry $g_{ab} + h^{\rm R}_{ab}$.

To avoid the singularity in h_{ab}, we replace the point particle abstraction by a small Schwarzschild black hole. The difficulty caused by the formal singularity is replaced by the requirement of boundary conditions at the event horizon. Following Mino, Sasaki and Tanaka [7], in section 6 we use a matched asymptotic expansion to demonstrate how the $\mathrm{O}(\mu)$ self-force adjusts the world line of the particle. For a small black hole moving in an external spacetime, the solution of the Einstein equations divides into two overlapping parts: In the *inner region* near the black hole the metric is approximately the Schwarzschild metric with a small perturbation caused by the external spacetime through which it is moving. In the *outer region* far from the black hole the metric is approximately the background geometry of the external spacetime with a small perturbation caused by the black hole. Let a length scale of the background be $\mathcal{R}$, and let r be some measure of distance from the black hole. Assume that $\mu \ll \mathcal{R}$ so that the black hole is in a context where it is meaningful to say that its mass is small. The inner region extends from the black hole out to $r \ll \mathcal{R}$. The outer region includes all $r \gg \mu$. These two regions overlap in the *buffer region* where $\mu \ll r \ll \mathcal{R}$.

When we focus on the inner region in sections 5 and 7 the object is a black hole, and we find an approximation for $h^{\rm S}$ that consists of the singular μ/r part of the Schwarzschild metric plus its tidal distortion caused by the background geometry. Equations (64)-(67) give a straightforward approximation for $h^{\rm S}_{ab}$. When we focus on the outer region we are free to think of the object as being a point particle. Matching the perturbed metrics in the "matching zone," within the buffer region, in section 6 provides an approximate solution to the Einstein equations with a remainder of $\mathrm{O}(\mu^2/\mathcal{R}^2)$, which is uniformly valid in the limit $\mu/\mathcal{R} \to 0$, everywhere outside the event horizon as is demonstrated in section 8.

The motion of the object is ultimately described as being geodesic in an abstract metric $g_{ab} + h^{\rm R}_{ab}$, where h^{R}_{ab} is the metric perturbation which would result from a point particle, with the singular part $h^{\rm S}_{ab}$ removed. The majority of the remainder of this manuscript is the elucidation of the steps which lead to the calculation of the $\mathrm{O}(\mu)$ adjustment of a small object's world line.

3. Locally inertial coordinate systems

A description of the metric perturbation h_{ab} near a point mass μ moving along a geodesic Γ is most convenient with coordinates in which the background geometry looks as flat as possible at the location of the particle. Let $\mathcal{R}$ be a representative length scale of the background geometry—the smallest of the radius of curvature, the scale of inhomogeneities, and the time scale for changes in curvature along Γ. Corresponding to any event p, there is always a *locally Minkowskii* coordinate system for which the metric and the affine connection at p are those of flat Minkowskii space, η_{ab}. The value of the metric and its first derivatives at p in any coordinate system are all that is required to determine a locally Minkowskii system. The construction is described, for example, by Weinberg [18] in his equation (3.2.12). Locally Minkowskii coordinates at p remain locally Minkowskii under an inhomogeneous Lorentz transformation. In addition, if p is the origin of the coordinates, then any transformation of the form $x^a_{\rm new} = x^a + \lambda^a{}_{bcd}x^b x^c x^d$ is also a locally Minkowskii system with the origin at p. Such an $\mathrm{O}(x^3)$ coordinate transformation changes the form of the metric only by $\mathrm{O}(x^2)$ in a neighborhood of p.

One generalization of locally Minkowskii coordinates, which fixes the form of the quadratic parts of the metric at p, are Riemann normal coordinates [19] where the metric takes the form

$$g_{ab} = \eta_{ab} - \frac{1}{6}(R_{acbd} - R_{adbc})x^c x^d + \mathrm{O}(x^3/\mathcal{R}^3). \tag{20}$$

Any coordinate transformation of the form

$$x^a_{\rm new} = x^a + \lambda^a{}_{bcde}x^b x^c x^d x^e + \mathrm{O}(x^5/\mathcal{R}^4) \tag{21}$$

preserves this Riemann normal form of the metric. The coordinate location of an event q is given in terms of a set of direction cosines, with respect to orthonormal basis vectors at p, and the change in affine parameter along a geodesic from p to q. Riemann normal coordinates are defined only in a region where the geodesics emanating from p do not intersect elsewhere in the region.

Coordinates $x^a = (t, x, y, z)$ may be found which are locally Minkowskii along any geodesic Γ, with t measuring the proper time s on Γ; such coordinates are said to be *locally inertial*. In these coordinates $g_{ab} = \eta_{ab} + \mathrm{O}(r^2/\mathcal{R}^2)$, where $r^2 \equiv x^2 + y^2 + z^2 \equiv x^i x_i$ and the indices i, j, k, l, p, q run over the spatial coordinates x, y and z. A coordinate transformation of the form $x^a_{\rm new} = x^a + \lambda^a{}_{ijk}(s)x^i x^j x^k + \mathrm{O}(r^4/\mathcal{R}^3)$ preserves these features with most components of the metric changing by $\mathrm{O}(r^2/\mathcal{R}^2)$. However, g_{tt} changes only by $\mathrm{O}(r^3/\mathcal{R}^3)$ and is always of the simple form $g_{tt} = -1 - R_{titj}x^i x^j + \mathrm{O}(r^3/\mathcal{R}^3)$, where R_{titj} is evaluated on Γ.

3.1. Fermi normal coordinates

Fermi normal coordinates [20] are one specialization of locally inertial coordinates on a geodesic Γ for which the $\mathrm{O}(r^2/\mathcal{R}^2)$ parts of the metric have a particularly appealing form as simple combinations of components of the Riemann tensor evaluated on Γ, [19]

$$\begin{aligned} g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = &-(1 + R_{titj}x^i x^j)\,\mathrm{d}t^2 - \frac{4}{3}R_{tikj}x^i x^j\,\mathrm{d}t\ \mathrm{d}x^k \\ &+ (f_{kl} - \frac{1}{3}R_{kilj}x^i x^j)\,\mathrm{d}x^k\,\mathrm{d}x^l \\ &+ \mathrm{O}(r^3/\mathcal{R}^3). \end{aligned} \tag{22}$$

Li and Ni [21] give the form of the metric in Fermi normal coordinates to higher order. The defining characteristics of Fermi normal coordinates are that they are orthogonal on Γ, that the spatial axes are geodesics, and that the distance from Γ at proper time s to an event $(t = s, x^i)$ is $(x^i x^j \delta_{ij})^{1/2}$, when measured along a geodesic perpendicular to Γ.

3.2. THZ Normal coordinates

A second specialization of locally inertial coordinates on Γ, introduced by Thorne and Hartle [22] and extended by Zhang [23], describe the external multipole moments, defined on Γ, of a vacuum solution of the Einstein equations. In these *THZ coordinates*

$$\begin{aligned} g_{ab} &= \eta_{ab} + H_{ab} \\ &= \eta_{ab} + {}_2H_{ab} + {}_3H_{ab} + \mathrm{O}(r^4/\mathcal{R}^4), \end{aligned} \tag{23}$$

with

$$\begin{aligned} {}_2H_{ab}dx^a dx^b = & -\mathcal{E}_{ij}x^i x^j (\,\mathrm{d}t^2 + f_{kl}\mathrm{d}x^k\,\mathrm{d}x^l) + \frac{4}{3}\epsilon_{kpq}\mathcal{B}^q{}_i x^p x^i \,\mathrm{d}t\,\mathrm{d}x^k \\ & - \frac{20}{21}\left[\dot{\mathcal{E}}_{ij}x^i x^j x_k - \frac{2}{5}r^2\dot{\mathcal{E}}_{ik}x^i\right]\mathrm{d}t\,\mathrm{d}x^k \\ & + \frac{5}{21}\left[x_i\epsilon_{jpq}\dot{\mathcal{B}}^q{}_k x^p x^k - \frac{1}{5}r^2\epsilon_{pqi}\dot{\mathcal{B}}_j{}^q x^p\right]\,\mathrm{d}x^i\,\mathrm{d}x^j + \mathrm{O}(r^4/\mathcal{R}^4) \end{aligned} \tag{24}$$

and

$$\begin{aligned} {}_3H_{ab}\mathrm{d}x^a\mathrm{d}x^b = & -\frac{1}{3}\mathcal{E}_{ijk}x^i x^j x^k(\,\mathrm{d}t^2 + f_{kl}\,\mathrm{d}x^k\,\mathrm{d}x^l) \\ & + \frac{2}{3}\epsilon_{kpq}\mathcal{B}^q{}_{ij}x^p x^i x^j\,\mathrm{d}t\,\mathrm{d}x^k + \mathrm{O}(r^4/\mathcal{R}^4), \end{aligned} \tag{25}$$

where ϵ_{ijk} is the flat space Levi-Civita tensor. These coordinates are well defined up to the addition of arbitrary functions of $\mathrm{O}(r^5/\mathcal{R}^4)$. The *external multipole moments* $\mathcal{E}_{ij}$, $\mathcal{B}_{ij}$, $\mathcal{E}_{ijk}$, and $\mathcal{B}_{ijk}$ are spatial, symmetric, tracefree (STF) tensors and are related to the Riemann tensor evaluated on Γ by

$$\mathcal{E}_{ij} = R_{titj}, \tag{26}$$

$$\mathcal{B}_{ij} = \epsilon_i{}^{pq}R_{pqjt}/2, \tag{27}$$

$$\mathcal{E}_{ijk} = [\partial_k R_{titj}]^{\mathrm{STF}} \tag{28}$$

and

$$\mathcal{B}_{ijk} = \frac{3}{8}\left[\epsilon_i{}^{pq}\partial_k R_{pqjt}\right]^{\mathrm{STF}}, \tag{29}$$

where $^{\mathrm{STF}}$ means to take the symmetric, tracefree part with respect to the spatial indices. $\mathcal{E}_{ij}$ and $\mathcal{B}_{ij}$ are $\mathrm{O}(1/\mathcal{R}^2)$, while $\mathcal{E}_{ijk}$ and $\mathcal{B}_{ijk}$ are $\mathrm{O}(1/\mathcal{R}^3)$. The dot denotes differentiation of the multipole moment with respect to t along Γ. Thus $\dot{\mathcal{E}}_{ij} = \mathrm{O}(1/\mathcal{R}^3)$ because $\mathcal{R}$ limits the time scale along Γ. All of the above external multipole moments are tracefree because the background geometry is assumed to be a vacuum solution of the Einstein equations.

The THZ coordinates are a specialization of harmonic coordinates, and it is useful to define the "Gothic" form of the metric

$$\mathfrak{g}^{ab} \equiv \sqrt{-g}g^{ab} \tag{30}$$

as well as

$$\bar{H}^{ab} \equiv \eta^{ab} - \mathfrak{g}^{ab}. \tag{31}$$

A coordinate system is harmonic if and only if

$$\partial_a \bar{H}^{ab} = 0. \tag{32}$$

Zhang [23] gives an expansion of $\mathfrak{g}^{ab}$ for an arbitrary solution of the vacuum Einstein equations in THZ coordinates, his equation (3.26). The terms of $\bar{H}^{ab}$ in this expansion include

$$\bar{H}^{ab} = {}_2\bar{H}^{ab} + {}_3\bar{H}^{ab} + \mathrm{O}(r^4/\mathcal{R}^4) \tag{33}$$

where

$$
\begin{aligned}
{}_2\bar{H}^{tt} &= -2\mathcal{E}_{ij}x^i x^j \\
{}_2\bar{H}^{tk} &= -\frac{2}{3}\epsilon^{kpq}\mathcal{B}_{qi}x_p x^i + \frac{10}{21}\left[\dot{\mathcal{E}}_{ij}x^i x^j x^k - \frac{2}{5}\dot{\mathcal{E}}_i{}^k x^i r^2\right] \\
{}_2\bar{H}^{ij} &= \frac{5}{21}\left[x^{(i}\epsilon^{j)pq}\dot{\mathcal{B}}_{qk}x_p x^k - \frac{1}{5}\epsilon^{pq(i}\dot{\mathcal{B}}^{j)}{}_q x_p r^2\right] \qquad (34)
\end{aligned}
$$

and

$$
\begin{aligned}
{}_3\bar{H}^{tt} &= -\frac{2}{3}\mathcal{E}_{ijk}x^i x^j x^k \\
{}_3\bar{H}^{tk} &= -\frac{1}{3}\epsilon^{kpq}\mathcal{B}_{qij}x_p x^i x^j \\
{}_3\bar{H}^{ij} &= \mathrm{O}(r^4/\mathcal{R}^4). \qquad (35)
\end{aligned}
$$

If $r/\mathcal{R} \ll 1$ then H_{ab} is approximately the trace reversed version of $\bar{H}^{ab}$,

$$
H_{ab} = \bar{H}_{ab} - \frac{1}{2}\eta_{ab}\bar{H}^c{}_c + \mathrm{O}(r^4/\mathcal{R}^4), \qquad (36)
$$

and (23)-(25) correspond precisely to (33)-(35) up to a remainder of $\mathrm{O}(r^4/\mathcal{R}^4)$.

Zhang [23] gives the transformation from Fermi normal coordinates to the THZ coordinates

$$
\begin{aligned}
t_{\mathrm{thz}} &= t_{\mathrm{fn}} \\
x^i_{\mathrm{thz}} &= x^i_{\mathrm{fn}} - \frac{r^2}{6}\mathcal{E}^i{}_j x^i_{\mathrm{fn}} + \frac{1}{3}\mathcal{E}_{jk}x^j_{\mathrm{fn}}x^k_{\mathrm{fn}}x^i_{\mathrm{fn}} + \mathrm{O}(r^4/\mathcal{R}^3). \qquad (37)
\end{aligned}
$$

3.3. An application of THZ coordinates

The scalar wave equation takes a particularly simple form in THZ coordinates,

$$
\begin{aligned}
\sqrt{-g}\nabla^a\nabla_a\psi &= \partial_a\big(\sqrt{-g}g^{ab}\partial_b\psi\big) \\
&= \partial_a\big(\eta^{ab}\partial_b\psi\big) - \partial_a\big(\bar{H}^{ab}\partial_b\psi\big) \\
&= (\eta^{ab} - \bar{H}^{ab})\partial_a\partial_b\psi, \qquad (38)
\end{aligned}
$$

where the second equality follows from (31) and the last from (32). After an expansion of the contractions on $\bar{H}^{ab}$, this becomes

$$
\sqrt{-g}\nabla^a\nabla_a\psi = \eta^{ab}\partial_a\partial_b\psi - \bar{H}^{ij}\partial_i\partial_j\psi - 2\bar{H}^{it}\partial_{(i}\partial_{t)}\psi - \bar{H}^{tt}\partial_t\partial_t\psi. \qquad (39)
$$

An approximate solution ψ with a point charge source is q/r. Direct substitution into (39) reveals just how good this approximation is. If ψ is replaced by q/r on the right hand side, then the first term gives a δ-function, the third and fourth terms vanish because r is independent of t, and in the second term ${}_2\bar{H}^{ij}$ has no contribution because of the details given in (34), and the $\mathrm{O}(r^4/\mathcal{R}^4)$ remainder of $\bar{H}^{ij}$ yields a term that scales as $\mathrm{O}(r/\mathcal{R}^4)$. Thus,

$$
\sqrt{-g}\nabla^a\nabla_a(q/r) = -4\pi q\delta^3(x^i) + \mathrm{O}(r/\mathcal{R}^4). \qquad (40)
$$

Note that the remainder $\mathrm{O}(r/\mathcal{R}^4)$ is C^0. From the consideration of solutions of Laplace's equation in flat spacetime, it follows that a C^2 correction to q/r, of $\mathrm{O}(r^3/\mathcal{R}^4)$, would remove the $\mathrm{O}(r/\mathcal{R}^4)$ remainder on the right hand side. We conclude that $q/r + \mathrm{O}(r^3/\mathcal{R}^4)$ is a solution of the scalar field wave equation for a point charge and that the error in the approximation of the solution by q/r is C^2. In Ref. [24] we show that q/r is the singular field ψ^{S} for a scalar charge, up to a remainder of $\mathrm{O}(r^3/\mathcal{R}^4)$. This was done by use of a Hadamard expansion of the Green's function.

THZ coordinates provide elementary, approximate solutions to the wave equation with a singular source for vector and tensor fields as well [25] .

4. Vector and tensor harmonics

The forms of ${}_2H_{ab}$ and ${}_3H_{ab}$ in (24) and (25) might appear unfamiliar, but they actually consist of $\ell = 2$ and 3 vector and tensor spherical harmonics and have a close relationship with those introduced by Regge and Wheeler [15] in their analysis of metric perturbations of Schwarzschild black holes. This relationship is clarified with an example of $\mathcal{E}_{ij}$, whose Cartesian components are symmetric, tracefree, and constant. However, the spherical-coordinate component $\mathcal{E}_{rr}$ has the angular dependence of a linear combination of the $Y_{\ell m}$'s for $\ell = 2$. Thus, it is convenient to define $\mathcal{E}^{(2)} \equiv \mathcal{E}_{ij}\hat{n}^i\hat{n}^j$, where $\hat{n}^i$ is the unit radial vector in flat space. $\mathcal{E}^{(2)}$ is a scalar field which carries all of the information contained in the constant Cartesian components of $\mathcal{E}_{ij}$ and may be used to generate related quadrupole vector and tensor harmonics.

For the angular components of vectors and tensors, we find it convenient to follow Thorne's description of the pure-spin vector and tensor harmonics [26], which are closely related to the harmonic decomposition used by Regge and Wheeler [15]. In the Schwarzschild geometry for example, the spin-1 vector harmonics generated by the spherical harmonic function $Y_{\ell m}$ are the even parity

$$Y_a^{\mathrm{E}\ell m} = r\sigma_a{}^b\nabla_b Y_{\ell m} \tag{41}$$

and the odd parity

$$Y_a^{\mathrm{B}\ell m} = -r\epsilon_a{}^b\nabla_b Y_{\ell m}, \tag{42}$$

where

$$\sigma_{ab} \equiv g_{ab} + u_a u_b - n_a n_b \tag{43}$$

is the metric of a constant t,r two-sphere, and

$$\epsilon_{ab} \equiv u^c n^d \epsilon_{cdab}, \quad \text{with} \quad \epsilon_{\theta\phi} = \epsilon_{tr\theta\phi} = r^2\sin\theta, \tag{44}$$

is the Levi-Civita tensor on the same two-sphere in the Schwarzschild geometry. Here u_a and n_a are the unit normals of surfaces of constant t and constant r, respectively.

We generalize this approach: For a vector field ξ_a, the parts $\sigma_a{}^b\xi_b$ which are tangent to a two-sphere may be described by two "potentials" ξ^{ev} and ξ^{od} via

$$\sigma_a{}^b\xi_b = r\sigma_a{}^b\nabla_b\xi^{\mathrm{ev}} - r\epsilon_a{}^b\nabla_b\xi^{\mathrm{od}}. \tag{45}$$

The potentials ξ^{ev} and ξ^{od} are generally functions of all of the spacetime coordinates and are guaranteed to exist by the invertibility of the two dimensional Laplacian on a two-sphere. The factors of r are included for convenience.

The notation for a covariant vector field is condensed by defining even and odd parity vectors associated with the potential ξ^{ev}

$$\xi_a^{\mathrm{ev}} \equiv r\sigma_a{}^b\nabla_b\xi^{\mathrm{ev}} \tag{46}$$

and with the potential ξ^{od}

$$\xi_a^{\mathrm{od}} \equiv -r\epsilon_a{}^b\nabla_b\xi^{\mathrm{od}}. \tag{47}$$

The four independent components of a covariant vector in a spherically symmetric geometry may be written as a sum of the form

$$\xi_a\,\mathrm{d}x^a = \xi_t\,\mathrm{d}t + \xi_r\,\mathrm{d}r + \left(\xi_A^{\mathrm{ev}} + \xi_A^{\mathrm{od}}\right)\,\mathrm{d}x^A \tag{48}$$

in terms of the four functions ξ_t, ξ_r, ξ^{ev} and ξ^{od}. The capital index A is used here just as a reminder that the vector to which it is attached is tangent to the two-sphere. The

A index should otherwise be considered an ordinary spacetime index in the covariant spirit of (45)-(47).

Similarly for a symmetric tensor field h_{ab}, the parts which are tangent to a two-sphere $\sigma_a{}^c\sigma_b{}^d h_{cd}$ may be described by the trace with respect to σ_{ab} and by two potentials h^{ev} and h^{od} via

$$\begin{aligned}\sigma_a{}^c\sigma_b{}^d h_{cd} = \frac{1}{2}h^{\mathrm{trc}}\sigma_{ab} + r^2\sigma_{(a}{}^c\sigma_{b)}{}^d\nabla_c\left(\sigma_d{}^e\nabla_e h^{\mathrm{ev}}\right) - \frac{1}{2}r^2\sigma_{ab}\sigma^{cd}\nabla_c\left(\sigma_d{}^e\nabla_e h^{\mathrm{ev}}\right)\\ - r^2\epsilon_{(a}{}^c\sigma_{b)}{}^d\nabla_c\left(\sigma_d{}^e\nabla_e h^{\mathrm{od}}\right)\end{aligned} \qquad (49)$$

The potentials h^{ev} and h^{od} are generally functions of all of the spacetime coordinates and are guaranteed to exist by theorems involving solutions of elliptic equations on a two-sphere. The factors of r^2 are included for convenience.

The notation for a covariant tensor field is condensed by defining trace-free tensors tangent to a two-sphere and associated with the potential h^{ev}

$$h^{\mathrm{ev}}_{ab} \equiv r^2\sigma_{(a}{}^c\sigma_{b)}{}^d\nabla_c\left(\sigma_d{}^e\nabla_e h^{\mathrm{ev}}\right) - \frac{1}{2}r^2\sigma_{ab}\sigma^{cd}\nabla_c\left(\sigma_d{}^e\nabla_e h^{\mathrm{ev}}\right) \qquad (50)$$

and with the potential h^{od}

$$h^{\mathrm{od}}_{ab} \equiv -r^2\epsilon_{(a}{}^c\sigma_{b)}{}^d\nabla_c\left(\sigma_d{}^e\nabla_e h^{\mathrm{od}}\right). \qquad (51)$$

The ten independent components of a symmetric covariant tensor h_{ab} in a spherically symmetric geometry may be written as a sum of the form

$$\begin{aligned}h_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = h_{tt}\,\mathrm{d}t^2 + 2h_{tr}\,\mathrm{d}t\,\mathrm{d}r + 2\left(h^{\mathrm{ev}}_{tA} + h^{\mathrm{od}}_{tA}\right)\,\mathrm{d}t\,\mathrm{d}x^A\\ + h_{rr}\,\mathrm{d}r^2 + 2\left(h^{\mathrm{ev}}_{rA} + h^{\mathrm{od}}_{rA}\right)\,\mathrm{d}r\,\mathrm{d}x^A\\ + \left(\frac{1}{2}h^{\mathrm{trc}}\sigma_{AB} + h^{\mathrm{ev}}_{AB} + h^{\mathrm{od}}_{AB}\right)\,\mathrm{d}x^A\,\mathrm{d}x^B\end{aligned} \qquad (52)$$

in terms of the ten functions h_{tt}, h_{tr}, h^{ev}_t, h^{od}_t, h_{rr}, h^{ev}_r, h^{od}_r, h^{trc}, h^{ev} and h^{od}, As with the vector field, the capital indices A and B are used here just as a reminder that the vector or tensor to which they are attached is tangent to the two-sphere. Otherwise, they should be considered ordinary spacetime indices in the covariant spirit of (49)-(51).

The descriptions of vector and tensor potentials in (45) and (49) on a two-sphere could have been written with a derivative operator involving the usual angular coordinates. However, this would cloud the covariant nature of the decomposition which is clearly revealed above.

The description of the vector and tensor components in terms of potentials takes advantage of the natural symmetry of the background geometry. For example, if a potential is a function of r and t times a $Y_{\ell m}$ then the resulting vector or tensor field is the same function times the vector or tensor spherical harmonic with the same ℓ, m pair. Expressions such as the perturbed Einstein tensor take a particularly simple form when written in terms of the potentials in place of the components.

We assume throughout that $\mathcal{E}$ is always associated with even parity vectors and tensors, and that $\mathcal{B}$ is always associated with odd parity vectors and tensors. Thus, $^{\mathrm{ev}}$ and $^{\mathrm{od}}$ are often understood in $\mathcal{E} = \mathcal{E}^{\mathrm{ev}}$ or $\mathcal{B} = \mathcal{B}^{\mathrm{od}}$. A superscript in parentheses, as in $\mathcal{E}^{(2)} = \mathcal{E}_{ij}n^in^j$, denotes the multipole index ℓ which is also the number of indices in the STF tensor $\mathcal{E}_{ij}$.

With this notation, alternative forms of (24) and (25) are

$$\begin{aligned}
{}_2H_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = &-r^2\mathcal{E}^{(2)}\left(\mathrm{d}t^2+\,\mathrm{d}r^2+\sigma_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^b\right)+2\frac{r^2}{3}\mathcal{B}^{(2)}_A\,\mathrm{d}t\,\mathrm{d}x^A\\
&-2\frac{2r^3}{7}\dot{\mathcal{E}}^{(2)}\,\mathrm{d}t\,\mathrm{d}r+2\frac{2r^3}{21}\dot{\mathcal{E}}^{(2)}_A\,\mathrm{d}t\,\mathrm{d}x^A\\
&+2\frac{r^3}{21}\dot{\mathcal{B}}^{(2)}_A\,\mathrm{d}r\,\mathrm{d}x^A-\frac{r^3}{42}\dot{\mathcal{B}}^{(2)}_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B+\mathrm{O}(r^4/\mathcal{R}^4) \quad (53)
\end{aligned}$$

and

$$\begin{aligned}
{}_3H_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = &-\frac{r^3}{3}\mathcal{E}^{(3)}(\,\mathrm{d}t^2+\,\mathrm{d}r^2+\sigma_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^b)\\
&+2\frac{r^3}{9}\mathcal{B}^{(3)}_A\,\mathrm{d}t\,\mathrm{d}x^A+\mathrm{O}(r^4/\mathcal{R}^4). \quad (54)
\end{aligned}$$

5. Slowly time dependent perturbations of the Schwarzschild geometry

When a small Schwarzschild black hole of mass μ moves through a background spacetime, the hole's metric is perturbed by tidal forces arising from H_{ab} in (23), and the actual metric near the black hole is

$$g^{\mathrm{act}}_{ab}=g^{\mathrm{Schw}}_{ab}+{}_2h_{ab}+{}_3h_{ab}+\mathrm{O}(r^4/\mathcal{R}^4), \quad (55)$$

where the quadrupole metric perturbation ${}_2h_{ab}$ is a solution of the perturbed Einstein equations (13). The appropriate boundary conditions for ${}_2h_{ab}$ are that it's components be well behaved on the future event-horizon, in a well-behaved coordinate system, and that ${}_2h_{ab}\to{}_2H_{ab}$ in the *buffer region* [22], where $\mu\ll r\ll\mathcal{R}$. The octupole metric perturbation ${}_3h_{ab}$ has a similar description.

In Appendix G we follow Poisson's recent analysis [27, 28, 29] of a tidally distorted black hole, and describe the metric perturbation for $r\ll\mathcal{R}$ in (G.6)-(G.11). An expansion of the metric perturbation in the buffer region for $\mu\ll r\ll\mathcal{R}$ ultimately provides the even parity

$$\begin{aligned}
{}_2h^{\mathrm{ev}}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = &-\mathcal{E}^{(2)}\left[(r-2\mu)^2\,\mathrm{d}t^2+r^2\,\mathrm{d}r^2+(r^2-2\mu^2)\sigma_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B\right]\\
&+\frac{16\mu^6}{15r^4}\dot{\mathcal{E}}^{(2)}\left[2(r+\mu)\,\mathrm{d}t^2+2(r+5\mu)\,\mathrm{d}r^2+(2r+5\mu)\sigma_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B\right]\\
&-2\frac{r(2r^3-3\mu r^2-6\mu^2r+6\mu^3)}{3(r-2\mu)}\dot{\mathcal{E}}^{(2)}\,\mathrm{d}t\,\mathrm{d}r+\mathrm{O}(\mu^8\dot{\mathcal{E}}^{(2)}/r^5), \quad (56)
\end{aligned}$$

and the odd parity

$$\begin{aligned}
{}_2h^{\mathrm{od}}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = &\,2\left[\frac{r}{3}(r-2\mu)\mathcal{B}^{(2)}_A+\frac{16\mu^6}{45r^4}(3r+4\mu)\dot{\mathcal{B}}^{(2)}_A\right]\mathrm{d}t\,\mathrm{d}x^A\\
&+2\frac{r^4}{12(r-2\mu)}\dot{\mathcal{B}}^{(2)}_A\,\mathrm{d}r\,\mathrm{d}x^A+\mathrm{O}(\mu^8\dot{\mathcal{B}}^{(2)}/r^5), \quad (57)
\end{aligned}$$

which together properly match the $\mathrm{O}(r^2/\mathcal{R}^2)$ terms of (24) or of (53); the $\mathrm{O}(r^3/\mathcal{R}^3)$ terms are in a different gauge. In this form $\mathcal{E}^{(2)}$ and $\mathcal{B}^{(2)}$ are considered functions of t, and $\dot{\mathcal{E}}^{(2)}$ denotes the t derivative of $\mathcal{E}^{(2)}$. Together, these provide the quadrupole metric perturbation up to remainders of $\mathrm{O}(r^4/\mathcal{R}^4)$ and $\mathrm{O}(\mu^8/r^5\mathcal{R}^3)$.

The approximately time independent octupole perturbation ${}_3H_{ab}$ of the small black hole may be treated similarly. The time independent solution of $E^{\mathrm{Schw}}_{ab}({}_3h)=0$

which is well behaved on the event horizon and properly matches the $\mathrm{O}(r^3/\mathcal{R}^3)$ terms of (25) or of (54)

$$
\begin{aligned}
{}_3h_{ab}dx^a\,\mathrm{d}x^b = &-\frac{r^3}{3}\mathcal{E}^{(3)}\left[\left(1-\frac{2\mu}{r}\right)^2\left(1-\frac{\mu}{r}\right)\mathrm{d}t^2\right.\\
&\left.+\left(1-\frac{\mu}{r}\right)\mathrm{d}r^2+\left(1-\frac{2\mu}{r}+\frac{4\mu^3}{5r^3}\right)\sigma_{AB}dx^A dx^B\right]\\
&+2\frac{r^3}{9}\left(1-\frac{2\mu}{r}\right)\left(1-\frac{4\mu}{3r}\right)\mathcal{B}^{(3)}_A\,\mathrm{d}t\,\mathrm{d}x^A. \qquad (58)
\end{aligned}
$$

The part of ${}_3h_{ab}$ proportional to $\dot{\mathcal{E}}_{ijk}$ or $\dot{\mathcal{B}}_{ijk}$ is of $\mathrm{O}(r^4/\mathcal{R}^4)$ and not required here.

At this level of approximation, the interactions of tidal forces with a small black hole have no significant effect upon the motion of the hole. From the analysis of Thorne and Hartle [22] the dominant tidal effect upon the motion of a nonrotating object results from the coupling between the external octupole moment of the geometry $\mathcal{E}_{ijk}$ and the internal quadrupole moment of the object $\mathcal{I}_{jk}$; the resulting force is

$$\mu a^i \sim \mathcal{E}^i{}_{jk}\mathcal{I}^{jk}, \qquad (59)$$

equation (1.12) of reference [22]. For a Schwarzschild black hole, $\mathcal{I}_{jk}$ must result from the external quadrupole moment $\mathcal{E}_{jk}$. With dimensional analysis we conclude that this tidal acceleration is no larger than

$$a^i \sim \mu^4\mathcal{E}^i{}_{jk}\mathcal{E}^{jk} \sim \mu^4/\mathcal{R}^5. \qquad (60)$$

This acceleration is much smaller than the $\mathrm{O}(\mu/\mathcal{R}^2)$ acceleration of the self-force which is the focus of this manuscript. Hence, we conclude that for our purposes the tidal forces resulting from (56)-(58) exert no significant net force on the black hole.

6. A small black hole moving through a background geometry

6.1. Buffer region

In the previous section we treated the actual metric of a small black hole moving through an external universe as the Schwarzschild metric being perturbed by tidal forces with a small perturbation parameter $r/\mathcal{R}$,

$$g^{\mathrm{act}}_{ab} = g^{\mathrm{Schw}}_{ab} + {}_2h_{ab} + {}_3h_{ab} + \mathrm{O}(r^4/\mathcal{R}^4), \qquad (61)$$

The metric perturbations ${}_2h_{ab}$ and ${}_3h_{ab}$ are the dominant perturbations arising from the quadrupole and octupole tidal forces and are given in (56)-(58).

In the buffer region $\mu \ll r \ll \mathcal{R}$ the actual metric is described equally well as the background metric being perturbed by the small mass μ with a perturbation parameter μ/r. With THZ coordinates the background metric is

$$g^0_{ab} = \eta_{ab} + {}_2H_{ab} + {}_3H_{ab} + \mathrm{O}(r^4/\mathcal{R}^4) \qquad (62)$$

and the actual metric is

$$g^{\mathrm{act}}_{ab} = g^0_{ab} + h^{\mu}_{ab} + h^{\mu^2}_{ab} + h^{\mu^3}_{ab} + \ldots \qquad (63)$$

Each $h^{\mu^n}_{ab}$ is the part of the metric perturbation which is proportional to μ^n. These are obtained by a re-expansion of the results of the previous section in terms of powers of the small parameter μ/r. Thus,

$$h^{\mu}_{ab} \equiv {}_0h^{\mu}_{ab} + {}_2h^{\mu}_{ab} + {}_3h^{\mu}_{ab} + (\mu r^3/\mathcal{R}^4), \qquad (64)$$

where

$$ {}_0h^{\mu}_{ab}dx^a\,\mathrm{d}x^b = 2\frac{\mu}{r}(\,\mathrm{d}t^2 + \,\mathrm{d}r^2) \tag{65} $$

is the μ/r part of the Schwarzschild metric g^{Schw}_{ab},

$$ \begin{aligned} {}_2h^{\mu}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = {} & 4\mu r\mathcal{E}^{(2)}\,\mathrm{d}t^2 - 2\frac{2\mu r}{3}\mathcal{B}^{(2)}_A\,\mathrm{d}t\,\mathrm{d}x^A \\ & + 2\frac{\mu r^2}{3}\dot{\mathcal{E}}^{(2)}\,\mathrm{d}t\,\mathrm{d}r + 2\frac{\mu r^2}{6}\dot{\mathcal{B}}^{(2)}_A\,\mathrm{d}r\,\mathrm{d}x^A \end{aligned} \tag{66} $$

consists of the $\mu r/\mathcal{R}^2$ and $\mu r^2/\mathcal{R}^3$ parts of ${}_2h_{ab}$ from (56) and (57), and ${}_3h^{\mu}_{ab}$ is the $\mu r^2/\mathcal{R}^3$ part of ${}_3h_{ab}$ in (58)

$$ \begin{aligned} {}_3h^{\mu}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = {} & \frac{\mu r^2}{3}\mathcal{E}^{(3)}\left[5\,\mathrm{d}t^2 + \,\mathrm{d}r^2 + 2\sigma_{AB}\mathrm{d}x^A\mathrm{d}x^B\right] \\ & - 2\frac{10\mu r^2}{27}\mathcal{B}^{(3)}_A\,\mathrm{d}t\,\mathrm{d}x^A. \end{aligned} \tag{67} $$

6.2. Asymptotic matching

To add a modest amount of formality to this analysis, we assume that the background metric g^0_{ab} with a geodesic Γ has an expansion in terms of THZ coordinates as in (62). We then consider a sequence of metrics $g_{ab}(\mu)$ which are solutions of the vacuum Einstein equations with a Schwarzschild black hole "centered on Γ" in the sense that near the black hole the metric is approximately described as in (61). The sequence is parameterized by $\mu \ll \mathcal{R}$ with $g_{ab}(0) = g^0_{ab}$. Our focus is on the behavior of $g_{ab}(\mu)$ in the limit that $\mu \to 0$. This analysis falls under the purview of singular perturbation theory [30]: $g_{ab}(\mu)$ has an event horizon if and only if $\mu \neq 0$; therefore, the exact metric for $\mu = 0$ differs fundamentally from a neighboring metric obtained in the limit $\mu \to 0$.

In the buffer region $g_{ab}(\mu)$ is nicely illustrated in a fashion introduced by Thorne and Hartle [22] as a sum of elements of positive powers of the small parameters μ/r and $r/\mathcal{R}$,

$$ \begin{array}{cccccccccccccc} g(\mu) & \sim & \eta & \& & 0 & \& & {}_2H' & \& & {}_3H' & \& & {}_4H' & \& \cdots = g^0 \\ & \& & \mu/r & \& & \mu/\mathcal{R} & \& & \mu r/\mathcal{R}^2 & \& & \mu r^2/\mathcal{R}^3 & \& & \mu r^3/\mathcal{R}^4 & \& \cdots = h^{\mu} \\ & \& & \mu^2/r^2 & \& & \mu^2/r\mathcal{R} & \& & \mu^2/\mathcal{R}^2 & \& & \mu^2 r/\mathcal{R}^3 & \& & \mu^2 r^2/\mathcal{R}^4 & \& \cdots = h^{\mu^2} \\ & \& & \mu^3/r^3 & \& & \mu^3/r^2\mathcal{R} & \& & \mu^3/r\mathcal{R}^2 & \& & \mu^3/\mathcal{R}^3 & \& & \mu^3 r/\mathcal{R}^4 & \& \cdots = h^{\mu^3} \\ & \& & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\ & & \overline{\quad g^{\mathrm{Schw}} \quad} & & \overline{\quad 0 \quad} & & \overline{\quad {}_2h' \quad} & & \overline{\quad {}_3h' \quad} & & \overline{\quad {}_4h' \quad} & \end{array} \tag{68} $$

where & means "and an element of the form ..." Starting with $\ell = 0$, the ℓth column in the tableau consists of elements which scale as $(r/\mathcal{R})^\ell$. Starting with $n = 0$, the nth row consists of elements which scale as $(\mu/r)^n$. In the $\mu/\mathcal{R} \to 0$ limit, every non-zero element in the tableau is larger than all elements below it in the same column, or to its right in the same row.

The primes on the H's in the top row work around a deficiency in our notation: In section 3.2 the prefix 2 in ${}_2H_{ab}$ refers to the multipole index $\ell = 2$. In the tableau, the prefix 2 on ${}_2H'_{ab}$ refers to the power of the order behavior, $\mathrm{O}(r^2/\mathcal{R}^2)$. While ${}_2H_{ab}$ includes not only the quadrupole parts proportional to $\mathcal{E}_{ij}$ and $\mathcal{B}_{ij}$, which are $\mathrm{O}(r^2/\mathcal{R}^2)$, but also the parts proportional to time derivatives of $\mathcal{E}_{ij}$ and $\mathcal{B}_{ij}$, which are the order of a higher power of $r/\mathcal{R}$. In the tableau, the time derivative terms of ${}_\ell H_{ab}$ are included in ${}_{\ell+1}H'_{ab}$ and columns further to the right.

Row n is proportional to μ^n and is an expansion in the external moments and in their time derivatives. Each element in the tableau is a *finite* combination of terms which scale with the same power of $1/\mathcal{R}$,

$$\mu^n r^{\ell-n}/\mathcal{R}^\ell \sim \left(\frac{\mu}{r}\right)^n r^\ell \left(\mathcal{E}_\ell \quad \& \quad {}^{(1)}\mathcal{E}_{\ell-1} \quad \& \quad {}^{(2)}\mathcal{E}_{\ell-1} \quad \& \quad \cdots \quad \& \quad {}^{(\ell-2)}\mathcal{E}_2\right) \qquad (69)$$

The prefix superscript is the number of time derivatives, and ${}^{(p)}\mathcal{E}_\ell$ represents the even or odd parity ℓ indexed STF external multipole moment differentiated with respect to time p times. Thus, ℓ is the largest external multipole index that contributes to any element in column ℓ or to ${}_\ell h'$.

At the outer edge of the buffer region, where $\mu/r \ll r/\mathcal{R}$, $g_{ab}(\mu)$ is approximately the background metric perturbed by μ. In this region, the top row of the tableau consists of the expansion of g^0_{ab} about Γ in powers of $r/\mathcal{R}$, contains no μ dependence and dominates the actual metric $g_{ab}(\mu)$. The sum of the top row is g^0_{ab}.

The $n = 1$ row combines to give h^μ_{ab} which is the $\mathrm{O}(\mu)$ metric perturbation of g^0_{ab}. And the nth row combines to give the $\mathrm{O}(\mu^n)$ perturbation; higher order perturbation theory for the background geometry is necessary to determine the $n > 1$ rows.

At the inner edge of the buffer region, where $\mu/r \gg r/\mathcal{R}$, $g_{ab}(\mu)$ is approximately the Schwarzschild geometry perturbed by background tidal forces. The $\ell = 0$ column of the tableau is simply an expansion of the Schwarzschild geometry in powers of μ/r, contains no $\mathcal{R}$ dependence and dominates the actual metric $g_{ab}(\mu)$.

The $\ell = 1$ column, linear in $r/\mathcal{R}$, would be a dipole perturbation of the Schwarzschild geometry. But there is no $r/\mathcal{R}$ term in an expansion about a geodesic. Consequently the top element of the $\ell = 1$ column is zero, as are all elements of this column.

The top term in the $\ell = 2$ column, ${}_2H'_{ab}$ represents the external quadrupole tidal field. When this is combined with the rest of the $\ell = 2$ column the result is ${}_2h'_{ab}$, the entire quadrupole perturbation of the black hole caused by tidal forces, in the time independent approximation. ${}_2h'_{ab}$ is given explicitly as in the $\mathrm{O}(1/\mathcal{R}^2)$ terms of (56) and (57).

Similarly, the top term in the $\ell = 3$ column, ${}_3H'_{ab}$ represents the $\mathrm{O}(r^3/\mathcal{R}^3)$ external tidal field which distorts the black hole creating ${}_3h'_{ab}$, which is given as the $\mathrm{O}(1/\mathcal{R}^3)$ terms in (56)-(58). Thus, the top element of each column provides a boundary condition for the equations which determine the resulting metric perturbation of the black hole. Each column also satisfies appropriate boundary conditions at the event horizon.

The analyses for ${}_\ell h'_{ab}$ up to $\ell = 3$ are straightforward problems in linear perturbation theory of a Schwarzschild black hole. The nonlinearity of the Einstein equations first appears in the elements of the $\ell = 4$ column, which have some contributions from terms quadratic in the $\ell = 2$ elements. Higher order perturbation theory for a black hole is necessary to determine the $\ell \geq 4$ columns.

The actual metric is accurately approximated by $g^{\mathrm{Schw}}_{ab} + {}_2h'_{ab} + {}_3h'_{ab}$ for $r \ll \mathcal{R}$, and g^0_{ab} is an accurate approximation of $g_{ab}(\mu)$ for $\mu \ll r$. In the buffer region $\mu \ll r \ll \mathcal{R}$ these approximations are

$$g^{\mathrm{Schw}}_{ab} + {}_2h'_{ab} + {}_3h'_{ab} = \eta_{ab} + {}_2H'_{ab} + {}_3H'_{ab} + \mathrm{O}(\mu/r) \qquad (70)$$

and

$$g^0_{ab} = \eta_{ab} + {}_2H'_{ab} + {}_3H'_{ab} + \mathrm{O}(r^4/\mathcal{R}^4). \qquad (71)$$

A demonstration of asymptotic matching [30] requires a *matching zone*, within the buffer region, where the smallest displayed term on the right hand side, ${}_3H'_{ab} = \mathrm{O}(r^3/\mathcal{R}^3)$, is simultaneously much larger than both remainder terms, $\mathrm{O}(\mu/r)$ and $\mathrm{O}(r^4/\mathcal{R}^4)$. The actual metric is accurately approximated by equation (70) to the "left" of the matching zone, by equation (71) to the "right" of the matching zone, and by $\eta_{ab} + {}_2H'_{ab} + {}_3H'_{ab}$ *only* within the matching zone.

The matching zone is thus bounded by $\mu/r \ll r^3/\mathcal{R}^3$ on the left and by $r^4/\mathcal{R}^4 \ll r^3/\mathcal{R}^3$ on the right. These may be combined into

$$(\mu\mathcal{R}^3)^{1/4} \ll r \ll \mathcal{R}, \tag{72}$$

and this fits within the buffer region because

$$\mu \ll (\mu\mathcal{R}^3)^{1/4} \ll r \ll \mathcal{R}, \qquad \mu/\mathcal{R} \to 0. \tag{73}$$

This is the signature of a matched asymptotic expansion.

7. Singular field h^{S}_{ab}

The Einstein tensor is the sum of terms consisting of the product of various components of the metric and its inverse along with two derivatives. In the buffer region, where $\mu \ll r \ll \mathcal{R}$, an expansion of the Einstein tensor $G_{ab}[g(\mu)]$ in positive powers of μ/r and $r/\mathcal{R}$ may be represented in a tableau similar to that for $g_{ab}(\mu)$ introduced in section 6.

In the expansion of $G_{ab}[g(\mu)]$ the terms of every power of $1/\mathcal{R}$ which contain no dependence upon μ are each zero because g^0_{ab} is assumed to be a vacuum solution of the Einstein equations, $G_{ab}(g^0) = 0$. Similarly, all of the terms linear in μ must combine to yield $E_{ab}(h^\mu) = -8\pi T_{ab}$, because h^μ_{ab} is a perturbative solution of the Einstein equations with T_{ab} representing a point mass. The individual terms in $g_{ab}(\mu)$ which are linear in μ also form an asymptotic expansion for h^μ; these are the ${}_\ell h^{\mu\prime}_{ab}$ terms in the $n = 1$ row of the tableau for $g_{ab}(\mu)$.

In sections 1 and 2 we discussed the actual metric perturbation h^{act}_{ab} from a point mass moving through an external geometry. The Hadamard form of the Green's function for the operator $E_{ab}(h)$ provides a decomposition $h^{\mathrm{act}}_{ab} = h^{\mathrm{S}}_{ab} + h^{\mathrm{R}}_{ab}$ in a neighborhood of Γ, where $E_{ab}(h^{\mathrm{S}}) = -8\pi T_{ab}$. The analysis of the Green's function yields an asymptotic expansion for h^{S}_{ab}. The remainder h^{R}_{ab} is necessarily a vacuum solution of $E_{ab}(h^{\mathrm{R}}) = 0$ in a neighborhood of Γ where an expansion for h^R_{ab} is regular. Thus, in the tableau for $g_{ab}(\mu)$, h^R_{ab} is $\mathrm{O}(\mu)$. However, its regular behavior in a neighborhood of Γ implies that it has no spatial dependence on a scale of $\mathrm{O}(\mu)$, and that it should properly be moved up in the tableau to be absorbed in the definition of g^0_{ab}. This $\mathrm{O}(\mu)$ change in g^0_{ab} would affect the $h^{\mu^n}_{ab}$ only for $n \geq 2$. Further, the actual constructions of ${}_2h'_{ab}$ and ${}_3h'_{ab}$, resulting in equations (56)-(58), do not appear to allow for the inclusion of any such regular part, except in the top row.

The possibility that h^{R}_{ab} when promoted to the top row, might contain a dipole part in the $\ell = 1$ column is discussed in section 8.

With no clear proof at hand, we thus provide the conjecture that the ${}_\ell h^{\mu\prime}_{ab}$ are the terms in an asymptotic expansion for h^{S}_{ab} and, therefore, that

$$h^{\mathrm{S}}_{ab} = h^\mu_{ab} \tag{74}$$

and that h^{R}_{ab} is included in the top row of the tableau (68). We have verified that ${}_0h^\mu_{ab}$ and ${}_2h^\mu_{ab}$ (in the Lorenz gauge) are equivalent to the first two terms in the expansion of

$h^{\rm S}_{ab}$ via the Hadamard form of the Green's function. Further, the ${}_\ell h^{\mu\prime}_{ab}$ have no dipole $\ell = 1$ component at $\mathrm{O}(\mu)$ which could effect the world line Γ at a level of interest in a self-force calculation.

In the next two sections the effect of coordinate choices on the form of $h^{\rm S}_{ab}$ are discussed. First, a change in the locally-inertial coordinates appears as a gauge transformation of the Schwarzschild metric being perturbed by the external tidal fields. Second, an $\mathrm{O}(\mu)$ coordinate change appears as a gauge transformation of the background metric being perturbed by a point mass μ.

7.1. Coordinate transformations of the locally inertial coordinates

The convenient THZ coordinate system is used in sections 5 and 6 to determine the leading terms ${}_0h^{\mu}_{ab}$, ${}_2h^{\mu\prime}_{ab}$ and ${}_3h^{\mu\prime}_{ab}$ in an expansion of $h^{\rm S}_{ab}$. But, if $h^{\rm S}_{ab}$ is to play a fundamental role in radiation reaction and self-force analyses then the definition of $h^{\rm S}_{ab}$ should certainly not be wed to any particular locally-inertial coordinate system.

In this subsection we examine the change in the description of $h^{\rm S}_{ab}$ under a change of locally-inertial coordinates. The next subsection describes how an $\mathrm{O}(\mu r^\ell/\mathcal{R}^\ell)$ gauge transformation of the perturbed Schwarzschild metric changes the form of $h^{\rm S}_{ab}$ while remaining with the same locally-inertial coordinates.

For the "inner" perturbation problem of the matched asymptotic expansions, the external tidal fields are considered a perturbation of the Schwarzschild geometry. From this perspective a change from one locally-inertial coordinate system to another appears as a gauge transformation of the perturbed Schwarzschild metric.

A second locally-inertial coordinate system is defined by

$$y^a = x^a + \lambda^a{}_{ijk}x^ix^jx^k + \mathrm{O}(r^4/\mathcal{R}^3), \tag{75}$$

where $\lambda^a{}_{ijk}$ is an $\mathrm{O}(1/\mathcal{R}^2)$ constant, such as in equation (37) which relates Fermi normal to THZ coordinates. For the perturbed Schwarzschild metric this appears as a gauge transformation with a gauge vector $\xi^a = \lambda^a{}_{ijk}x^ix^jx^k + \mathrm{O}(r^4/\mathcal{R}^3)$. Under such a change in coordinates the description of $h^{\rm S}_{ab}$ changes in two different ways: the functional dependence upon coordinate position changes and the components of the tensor change. Let the components in the y coordinate system be denoted by a prime. For a fixed coordinate position κ^c,

$$h^{\rm S}_{a'b'}\big|_{y^c=\kappa^c} = \left(h^{\rm S}_{ab}\big|_{x^c=\kappa^c} - \xi^c\partial_c h^{\rm S}_{ab}\right)\frac{\partial x^a}{\partial y^{a'}}\frac{\partial x^b}{\partial y^{b'}} + \mathrm{O}(\mu r^2/\mathcal{R}^3), \tag{76}$$

which, when expanded out, is

$$h^{\rm S}_{a'b'} = h^{\rm S}_{ab} - \xi^c\partial_c h^{\rm S}_{ab} - 2h^{\rm S}_{c(a}\partial_{b)}\xi^c + \mathrm{O}(\mu r^2/\mathcal{R}^3). \tag{77}$$

The left hand side is evaluated at $y^c = \kappa^c$ and the right hand side at $x^c = \kappa^c$. In terms of the Lie derivative $\pounds$, the descriptions of the single tensor field $h^{\rm S}_{ab}$ in two different locally-inertial coordinate systems are related by

$$h^S_{a'b'} = h^S_{ab} - \pounds_\xi\, h^{\rm S}_{ab} + \mathrm{O}(\mu r^2/\mathcal{R}^3). \tag{78}$$

Now, $h^{\rm S}_{ab} = {}_0h^{\mu}_{ab} + {}_2h^{\mu}_{ab} + \mathrm{O}(\mu r^2/\mathcal{R}^3)$, as in (64), and ${}_0h^{\mu}_{ab} = \mathrm{O}(\mu/r)$ in any locally-inertial coordinates. Thus, the change in $h^{\rm S}_{ab}$ is most naturally assigned to ${}_2h^{\mu}_{ab}$,

$${}_2h^{\mu\rm new}_{ab} = {}_2h^{\mu\rm old}_{ab} - \pounds_\xi\, {}_0h^{\mu}_{ab} + \mathrm{O}(\mu r^2/\mathcal{R}^3). \tag{79}$$

This description of the change in the ${}_2h^{\mu}_{ab}$ part of $h^{\rm S}_{ab}$ is consistent with the related gauge transformation of the $\ell = 2$ metric perturbation, ${}_2h_{ab} = {}_2H_{ab} + {}_2h^{\mu}_{ab} + {\rm O}(\mu^2/\mathcal{R}^2)$, of the Schwarzschild geometry

$$ {}_2h^{\rm new}_{ab} = {}_2h^{\rm old}_{ab} - \pounds_\xi g^{\rm Schw}_{ab} + {\rm O}(r^3/\mathcal{R}^3). \tag{80} $$

The leading terms of this for large r are

$$ {}_2H^{\rm new}_{ab} + {}_2h^{\mu{\rm new}}_{ab} = {}_2H^{\rm old}_{ab} + {}_2h^{\mu{\rm old}}_{ab} - \pounds_\xi(\eta_{ab} + {}_0h^{\mu}_{ab}) + {\rm O}(r^3/\mathcal{R}^3,\, \mu r^2/\mathcal{R}^3). \tag{81} $$

These are naturally apportioned as

$$ \begin{aligned} {}_2H^{\rm new}_{ab} &= {}_2H^{\rm old}_{ab} - \pounds_\xi \eta_{ab} + {\rm O}(r^3/\mathcal{R}^3) \\ &= {}_2H^{\rm old}_{ab} - 2\nabla_{(a}\xi_{b)} + {\rm O}(r^3/\mathcal{R}^3) \end{aligned} \tag{82} $$

and

$$ {}_2h^{\mu{\rm new}}_{ab} = {}_2h^{\mu{\rm old}}_{ab} - \pounds_\xi\, {}_0h^{\mu}_{ab} + {\rm O}(\mu r^2/\mathcal{R}^3). \tag{83} $$

A comparison of (79) and (83) reveals the consistency of the description of $h^{\rm S}_{ab}$ as a single tensor field, which in any normal coordinate system is approximated by ${}_0h^{\mu}_{ab} + {}_2h^{\mu}_{ab} + {}_3h^{\mu}_{ab} + {\rm O}(\mu r^3/\mathcal{R}^4)$ for $\mu \ll r \ll \mathcal{R}$.

An ${\rm O}(r^4/\mathcal{R}^3)$ transformation changes ${}_3h^{\mu}_{ab}$ in a similar way.

7.2. Transformation of h^S_{ab} to the Lorenz gauge

The convenient Regge-Wheeler gauge was used, with the THZ coordinates, in sections 5 and 6 to determine the leading terms ${}_0h^{\mu}_{ab}$, ${}_2h^{\mu\prime}_{ab}$ and ${}_3h^{\mu\prime}_{ab}$ in an expansion of $h^{\rm S}_{ab}$. But, if $h^{\rm S}_{ab}$ is to play a fundamental role in radiation reaction and self-force analyses then the definition of $h^{\rm S}_{ab}$ should certainly not be wed to any particular gauge choice.

This section gives an example of an ${\rm O}(\mu r^\ell/\mathcal{R}^\ell)$ gauge transformation, for $\ell = 0$ and 2, of the perturbed background metric g^0_{ab} which changes the form of $h^{\rm S}_{ab}$ while remaining with the same locally-inertial coordinates. The previous subsection describes how the description of $h^{\rm S}_{ab}$ changes under a change of locally-inertial coordinates.

$h^{\rm S}_{ab}$ is given above in (64)-(67) in the Regge-Wheeler gauge. To transform the ${}_2h^{\mu\prime}_{ab}$ part of this into the Lorenz gauge, the gauge vector is

$$ \xi^a = -\mu(1 - r^2\mathcal{E}^{(2)})\delta^a_r + \mu r^2 \mathcal{E}^{(2)}_B \sigma^{Ba}. \tag{84} $$

The Lorenz gauge has

$$ {}_2h^{\mu\prime}_{ab}({\rm lz}) = {}_2h^{\mu\prime}_{ab}({\rm rw}) - \nabla_a\xi_b - \nabla_b\xi_a, \tag{85} $$

where the metric being perturbed is g^0_{ab}, and ∇_a is its covariant derivative operator. This results in

$$ \begin{aligned} {}_2h^{\mu\prime}_{ab}({\rm lz})\, {\rm d}x^a\, {\rm d}x^b = {} & \frac{2\mu}{r}\Big[(1 + r^2\mathcal{E}^{(2)})\, {\rm d}t^2 + (1 - 3r^2\mathcal{E}^{(2)})\, {\rm d}r^2 + (1 - r^2\mathcal{E}^{(2)})\sigma_{AB}\, {\rm d}x^A\, {\rm d}x^B\Big] \\ & - 4\mu r\mathcal{E}^{(2)}_A\, {\rm d}r\, {\rm d}x^A - 2\mu r\mathcal{E}^{(2)}_{AB}\, {\rm d}x^A\, {\rm d}x^B \\ & + 2\frac{\mu}{3r}\mathcal{B}^{(2)}_A\, {\rm d}t\, {\rm d}x^A + {\rm O}(\mu r^2/\mathcal{R}^3). \end{aligned} \tag{86} $$

For completeness, the trace of ${}_2h^{\mu\prime}_{ab}({\rm lz})$ is

$$ (\eta^{ab} - {}_2H^{ab})\,{}_2h^{\mu\prime}_{ab}({\rm lz}) = 4\mu/r + {\rm O}(\mu r^2/\mathcal{R}^3), \tag{87} $$

and the trace-reversed ${}_2\bar{h}^{\mu\prime}_{ab} \equiv {}_2h^{\mu\prime}_{ab} - \frac{1}{2} g^0_{ab} g_0^{cd} {}_2h^{\mu\prime}_{cd}$ is

$$
\begin{aligned}
{}_2\bar{h}^{\mu\prime}_{ab}(\mathrm{lz})\, \mathrm{d}x^a\, \mathrm{d}x^b = {}& \frac{4\mu}{r}(1 + r^2\mathcal{E}^{(2)})\, \mathrm{d}t^2 - 4\mu r \mathcal{E}^{(2)}\, \mathrm{d}r^2 \\
& - 4\mu r \mathcal{E}^{(2)}_A\, \mathrm{d}r\, \mathrm{d}x^A - 2\mu r \mathcal{E}^{(2)}_{AB}\, \mathrm{d}x^A\, \mathrm{d}x^B \\
& - 2\frac{r\mu}{3}\mathcal{B}^{(2)}_A\, \mathrm{d}t\, \mathrm{d}x^A,
\end{aligned} \tag{88}
$$

which satisfies the Lorenz gauge condition.

$$
\nabla^a {}_2\bar{h}^{\mu\prime}_{ab}(\mathrm{lz}) = \mathrm{O}(\mu r/\mathcal{R}^3). \tag{89}
$$

Equation (86) gives ${}_2h^{\mu\prime}_{ab}$ in the Lorenz gauge with THZ coordinates. From the perspective of the background metric g^0_{ab}, a change from THZ to Fermi normal coordinates, as described in the previous section, would preserve the covariant condition (89) for the Lorenz gauge and provide $h^{\mathrm{S}}_{ab}(\mathrm{lz})$ in Fermi normal coordinates.

8. Regular field $h^{\mathbf{R}}_{ab}$

In a self-force application, it is first required to find the actual metric perturbation h^{act}_{ab} for a point mass μ moving along a geodesic Γ of the background spacetime g^0_{ab}. In many cases h^{act}_{ab} will be the retarded metric perturbation. However, we prefer to leave the choice of boundary conditions general.

From the expansion of g^0_{ab} about Γ, as in 3, the first few terms of an asymptotic expansion for h^{S}_{ab} is determined as in 6. The *regular remainder* is defined by

$$
h^{\mathrm{R}}_{ab} \equiv h^{\mathrm{act}}_{ab} - h^{\mathrm{S}}_{ab} \tag{90}
$$

in a neighborhood of Γ where $E_{ab}(h^{\mathrm{R}}) = 0$. h^R_{ab} does not change over an $\mathrm{O}(\mu)$ length scale, so it is natural to combine h^R_{ab} with g^0_{ab} in the top row of the tableau of 6. Then the condition that the dipole term of the top row is zero is equivalent to the condition that Γ is actually a geodesic of $g^0_{ab} + h^{\mathrm{R}}_{ab}$.

From a different perspective, if h^R_{ab} is left in the μ^1 row, and if Γ is not a geodesic of $g^0_{ab} + h^{\mathrm{R}}_{ab}$, then h^{R}_{ab} necessarily has a dipole part in its expansion about Γ. This implies that the gravitational field of μ is not centered upon Γ. The act of adjusting Γ to remove the dipole field and to accurately track the center of the gravitational field of μ is equivalent to requiring that Γ be a geodesic of $g^0_{ab} + h^{\mathrm{R}}_{ab}$. This act of adjustment is also equivalent to performing a gauge transformation to the perturbed Schwarzschild geometry that removes the dipole field.

Thus the consistency of the matched asymptotic expansions implies that, indeed, the $\mathrm{O}(\mu)$ correction to geodesic motion for an infinitesimal black hole has the motion being geodesic in $g^0_{ab} + h^{\mathrm{R}}_{ab}$.

In an actual calculation, an exact expression for h^{S}_{ab} is usually not available. It is only necessary that $g^0_{ab} + h^{\mathrm{R}}_{ab}$ be C^1 in order to determine a geodesic, and this C^1 requirement can be met as long as the approximation for h^{S}_{ab} includes at least ${}_0h^{\mu\prime} + {}_2h^{\mu\prime}$. The next term is ${}_3h^{\mu\prime} = \mathrm{O}(\mu r^2/\mathcal{R}^3)$, and its derivative necessarily vanishes on Γ where $r = 0$. Thus, calculations of the self-force will be successful as long as the monopole and quadrupole terms of the asymptotic expansion for h^{S}_{ab} are included in the evaluation of h^{R}_{ab} via (90). Nevertheless, including the higher order terms of h^{S}_{ab}, results in the approximation for h^{R}_{ab} being more differentiable. In a calculation, usually h^{act}_{ab} is determined as a sum over modes with h^{R}_{ab} being decomposed in terms of the same modes. In determining the self-force, the more differentiable the h^{R}_{ab} is, the more rapidly the sum over modes converges. The use of higher order terms in an approximation for $h^{\rm S}_{ab}$ can have dramatic effects on the ultimate accuracy of a self-force calculation [24].

If we have the actual $\mathrm{O}(\mu)$ metric perturbation $h^{\rm act}_{ab}$ for a point mass, then the asymptotic matching provides an approximation for the geometry of a small black hole moving in the external geometry $g_{ab}(\mu)$ in the limit that $\mu/\mathcal{R} \to 0$. The approximation extends throughout the entire external spacetime down to the event horizon. Further, the approximation is revealed to be uniformly valid by the concise description of the matched geometry as

$$\begin{aligned} g_{ab}(\mu) = {} & (g^0_{ab} + h^{\rm act}_{ab}) + (g^{\rm Schw}_{ab} + {}_2h'_{ab} + {}_3h'_{ab}) \\ & - (\eta_{ab} + {}_2H'_{ab} + {}_3H'_{ab} + {}_0h^{\mu\prime}_{ab} + {}_2h^{\mu\prime}_{ab} + {}_3h^{\mu\prime}_{ab}) + \mathrm{O}(\mu^2/\mathcal{R}^2), \;\; \mu/\mathcal{R} \to 0. \end{aligned} \tag{91}$$

The combination $g^0_{ab} + h^{\rm act}_{ab}$ includes all terms in the top two rows of the tableau but extends outside the buffer region to include the entire external spacetime. The combination $g^{\rm Schw}_{ab} + {}_2h'_{ab} + {}_3h'_{ab}$ is the left four columns in the tableau. The remaining terms keep the entire expression from double-counting the elements in the upper left corner. The dominant term from the tableau which is not included here is $\mathrm{O}(\mu^2 r^2/\mathcal{R}^4)$ which gives the $\mathrm{O}(\mu^2/\mathcal{R}^2)$ remainder for this uniformly valid approximation for the matched metric in the limit $\mu/\mathcal{R} \to 0$.

9. Gauge issues

9.1. Gauge transformations

In perturbation analyses of general relativity [31, 32, 33], one considers the difference in the actual metric $g^{\rm act}_{ab}$ of an interesting, perturbed spacetime and the abstract metric g^0_{ab} of some given, background spacetime. The difference

$$h_{ab} = g^{\rm act}_{ab} - g^0_{ab} \tag{92}$$

is assumed to be infinitesimal, say $\mathrm{O}(h)$. Typically, one determines a set of linear equations which govern h_{ab} by expanding the Einstein equations through $\mathrm{O}(h)$. The results are often used to resolve interesting issues concerning the stability of the background, or the propagation and emission of gravitational waves by a perturbing source.

However, (92) is ambiguous: The metrics $g^{\rm act}_{ab}$ and g^0_{ab} are given on different manifolds. For a given event on one manifold at which corresponding event on the other manifold is the subtraction to be performed? Usually a coordinate system common to both spacetimes induces an implicit mapping between the manifolds and defines the subtraction. Yet, the presence of the perturbation allows ambiguity. An infinitesimal coordinate transformation of the perturbed spacetime

$$x'^a = x^a + \xi^a, \quad \text{where} \quad \xi^a = \mathrm{O}(h), \tag{93}$$

not only changes the components of a tensor at $\mathrm{O}(h)$, in the usual way, but also changes the mapping between the two manifolds in (92). After the transformation (93),

$$h^{\rm new}_{ab} = \left(g^0_{cd} + h^{\rm old}_{cd}\right) \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} - \left(g^0_{ab} + \xi^c \frac{\partial g^0_{ab}}{\partial x^c}\right). \tag{94}$$

The ξ^c in the last term accounts for the O(h) change in the event of the background used in the subtraction. After an expansion, this provides a new description of h_{ab}

$$\begin{aligned} h^{\rm new}_{ab} &= h^{\rm old}_{ab} - g^0_{cb}\frac{\partial \xi^c}{\partial x^a} - g^0_{cb}\frac{\partial \xi^d}{\partial x^b} - \xi^c\frac{\partial g^0_{ab}}{\partial x^c} \\ &= h^{\rm old}_{ab} - \pounds_\xi g^0_{ab} = h^{\rm old}_{ab} - 2\nabla_{(a}\xi_{b)} \end{aligned} \qquad (95)$$

through O(h); the symbol $\pounds$ represents the Lie derivative and ∇_a is the covariant derivative compatible with g^0_{ab}.

The action of such an infinitesimal coordinate transformation is called a *gauge transformation* and does not change the actual perturbed manifold, but it does change the coordinate description of the perturbed manifold.

A similar circumstance holds with general coordinate transformations. A change in coordinate system creates a change in description. But, general covariance dictates that actual physical measurements must be describable in a manner which is invariant under a change in coordinates. Thus, one usually describes physically interesting quantities strictly in terms of geometrical scalars which, by nature, are coordinate independent.

In a perturbation analysis any physically interesting result ought to be describable in a manner which is gauge invariant.

9.2. Gauge invariant quantities

Gauge invariant quantities appear to fall into a few different categories.

The change in any geometrical quantity under a gauge transformation is determined by the Lie derivative of that same quantity on the background manifold. This is demonstrated for the gauge transformation of a metric perturbation in (95). We also used this fact to describe the change in $h^{\rm S}_{ab}$ under gauge transformations in sections 7.1 and 7.2. Thus, if a geometrical quantity vanishes in the background, but not in the perturbed metric, then it will be gauge invariant. Examples include the Newman-Penrose scalars Ψ_0 and Ψ_4 which vanish for the Kerr metric. In the perturbed Kerr metric Ψ_0 and Ψ_4 are non zero, gauge invariant and the basis for perturbation analyses of rotating black holes. A second example has the background metric being a vacuum solution of the Einstein equations, so its Ricci tensor R_{ab} vanishes. The Ricci tensor of a perturbation of this metric is then unchanged by a gauge transformation. This is directly demonstrated in Appendix D.

Some quantities which are associated with a symmetry of the perturbed geometry are gauge invariant. For example a geodesic of a perturbed Schwarzschild metric, where the perturbation is axisymmetric with Killing field k^a, has a constant of motion $k^a u^b(g^0_{ab} + h_{ab})$ which is gauge invariant.

Another symmetry example involves the Schwarzschild geometry with an arbitrary perturbation. It is a fact that a gauge transformation can always be found, such that the resulting h_{ab} has the components $h_{\theta\theta}$, $h_{\theta\phi}$ and $h_{\phi\phi}$ all equal to zero. In this gauge, the surfaces of constant r and t are geometrical two-spheres, even while the manifold as whole has no symmetry. The area of each two-sphere can be used to define a radial scalar field R which is constant on each of these two-spheres. This scalar field on the perturbed Schwarzschild manifold is independent of gauge. However, its coordinate description in terms of the usual t, r, θ, ϕ coordinates does change under a gauge transformation. We find a use for this gauge invariant scalar field in section 10.3.

Quantities which are carefully described by a physical measurement are gauge invariant. For example, the acceleration of a world line could be measured with masses and springs by an observer moving along a world line in a perturbed geometry. The magnitude of the acceleration is a scalar and is gauge invariant. If the world line has zero acceleration, then it is a geodesic. Therefore, a geodesic of a perturbed metric remains a geodesic under a gauge transformation even while its coordinate description changes by $\mathrm{O}(h)$.

The mass and angular momentum are other gauge invariant quantities which might be measured by distant observers in an asymptotically flat spacetime. A small mass orbiting a larger black hole perturbs the black hole metric and emits gravitational waves. The gravitational waveform measured at a large distance is also gauge invariant.

9.3. Gauge transformations and the gravitational self-force

We understand that a point mass moves along a world line of a background metric g^0_{ab} and causes a metric perturbation h^{act}_{ab}, which may be decomposed into h^{S}_{ab} and h^{R}_{ab}. The gravitational self-force makes the world line be a geodesic of $g^0_{ab} + h^{\mathrm{R}}_{ab}$. This world line is equivalently described in terms of the background metric and its perturbation by

$$u^b\nabla_b u^a = -(g_0^{ab} + u^a u^b)u^c u^d(\nabla_c h^{\mathrm{R}}_{db} - \frac{1}{2}\nabla_b h^{\mathrm{R}}_{cd}) = \mathrm{O}(\mu/\mathcal{R}^2) \qquad (96)$$

where the covariant derivative and normalization of u^a are compatible with g^0_{ab}.

Given this world line, let ξ^a be a differentiable vector field which is equal, on the world line, to the $\mathrm{O}(\mu)$ displacement back to the geodesic of g^0_{ab} along which the particle would move in the absence of h^{R}_{ab}; otherwise ξ^a is arbitrary. Such a ξ^a generates a gauge transformation for which the right hand side of (96) is zero when evaluated with the new h_{ab} [1]. With the new h_{ab} there is no "gravitational self-force", and the coordinate description of the world line is identical to the coordinate description of a geodesic of g^0_{ab}. With or without the gauge transformation, an observer moving along this world line would measure no acceleration and would conclude that the world line is a geodesic of the perturbed metric.

This example shows that simple knowledge of the gravitational self-force, as defined in terms of the right hand side of (96), is not a complete description of any physically interesting quantity.

With this same example, after a time T the gauge vector $\xi \sim T^2\dot{u} \sim T^2\mu/\mathcal{R}^2$, and as long as $T \lesssim \mathcal{R}$ the gauge vector ξ^a remains small. However, when $T \sim \mathcal{R}\sqrt{\mathcal{R}/\mu}$, which is much larger than the dynamical timescale $\mathcal{R}$, the gauge vector $\xi \sim \mathcal{R}$ and can no longer be considered small. Thus, a gauge choice which cancels the coordinate description of the self-force necessarily fails after a sufficiently long time. Mino [34] takes advantage of this fact in his proposal to find the cumulative, gravitational self-force effect on the Carter constant for eccentric orbits around a rotating black hole.

10. An example: self force on circular orbits of the Schwarzschild metric

The introduction described the Newtonian problem of a small mass μ in a circular orbit of radius R about a much larger mass M. The analysis results in the usual $\mathrm{O}(\mu/M)$ reduced mass effect on the orbital frequency Ω given in (2). Reference [2] has a thorough introduction to the mechanics of this self-force calculation and gives a detailed discussion of this elementary problem using the same language and style which is common for the relativistic gravitational self-force. This includes elementary expressions for the S and R-fields of the Newtonian gravitational potential with descriptions of their decompositions in terms of spherical harmonics.

The extension of this Newtonian problem to general relativity is perhaps the simplest, interesting example of the relativistic gravitational self-force. Thus, we focus on a small mass μ in a circular geodesic about a Schwarzschild black hole of mass M, and we describe each of the steps necessary to obtain physically interesting results related to the gravitational self-force.

10.1. Mode sum analysis

Metric perturbations of Schwarzschild have been thoroughly studied since Regge and Wheeler [15, 17]. Both T_{ab} and h_{ab} are fourier analyzed in time, with frequency ω, and decomposed in terms of tensor spherical harmonics, with multipole indices ℓ and m. Linear combinations of the components of $h_{ab}^{\ell m,\omega}$ satisfy elementary ordinary differential equations which are easily numerically integrated. With the periodicity of a circular orbit, only a discrete set frequencies $\omega_m = -m\Omega$ appear.

We assume, then, that $h_{ab}^{\ell m,\omega}(r)$ can be determined for any ℓ and m. The sum of these over all ℓ and m then constitutes h_{ab}^{act}, and this sum will be divergent if evaluated at the location of μ.

The next task is to determine h_{ab}^{S}. The THZ coordinates, including $\mathrm{O}(r^4/\mathcal{R}^3)$ terms, for a circular orbit of Schwarzschild are given in reference [24]. Equations (65)-(67) give an approximation for h_{ab}^{S} in THZ coordinates with a remainder of $\mathrm{O}(\mu r^3/\mathcal{R}^4)$.

We follow the mode-sum regularization procedure pioneered by Barack and Ori [35, 36, 37, 38] and Mino, Sasaki and Tanaka [39, 40] and followed up by others [24, 25, 41, 42]. In this procedure, the multipole moments of the S-field are calculated and referred to as *regularization parameters*. The sum of these moments diverges when evaluated at the location of μ, but each individual moment is finite. Importantly, the S-field has been constructed to have precisely the same singularity structure at the particle as the actual field has. Thus the difference in these moments gives a multipole decomposition of the regular R-field. Schematically, this procedure gives

$$h_{ab}^{\mathrm{R}} = \sum_{\ell m,\omega} h_{ab}^{\mathrm{R}(\ell m,\omega)} = \sum_{\ell m,\omega} \left[h_{ab}^{\mathrm{act}(\ell m,\omega)} - h_{ab}^{\mathrm{S}(\ell m,\omega)} \right] \tag{97}$$

for the regular field.

We note that the sum over modes of a decomposition of a C^∞ function converges faster than any power of ℓ. And, the less the differentiability of the function then the slower the convergence of its mode sum. Exact values for $h_{ab}^{\mathrm{S}(\ell m,\omega)}$ would then give rapid convergence of the sum yielding a C^∞ representation of h_{ab}^{R}. However, the approximation for h_{ab}^{S} in (65)-(67) has an $\mathrm{O}(\mu r^3/\mathcal{R}^4)$ remainder which is necessarily only C^2. This immediately puts a limitation on the rate of convergence of any mode sum for h_{ab}^{R}. Further, h_{ab}^{S} is only defined in a neighborhood of μ. Whereas a decomposition in terms of spherical harmonics requires a field defined over an entire two-sphere. It is important that the extension of h_{ab}^{S} over the two-sphere is C^∞ everywhere, except at μ, to insure rapid convergence of the mode sum. This ambiguity for h_{ab}^{S}, away from μ, highlights an important fact: the value of any individual multiple moment $h_{ab}^{\mathrm{S}(\ell m,\omega)}$ or $h_{ab}^{\mathrm{R}(\ell m,\omega)}$ is inherently ill defined. Only a sum over modes, such as in (97), might have physical meaning.

10.2. Gauge issues

A vexing difficulty with equation (97) revolves around gauge transformations. What assurance do we have that the singularity structure of h^{S}_{ab} truly matches the singularity structure of h^{act}_{ab}? For example h^{S}_{ab} is often described in the Lorenz gauge which is well behaved by most standards, whereas h^{act}_{ab} is most easily calculated in the Regge-Wheeler gauge which often entails discontinuities in components of h_{ab}.

A gauge transformation does not change the relationship

$$E_{ab}(h^{\mathrm{act}}) = E_{ab}(h^{\mathrm{S}}) + E_{ab}(h^{\mathrm{R}}) = -8\pi T_{ab}. \tag{98}$$

But it also does not dictate how to apportion a gauge transformation for h^{act}_{ab} between h^{S}_{ab} and h^{R}_{ab}. In a neighborhood of the particle h^{R}_{ab} is known to be a solution of $E_{ab}(h^{\mathrm{R}}) = 0$, but a gauge transformation generates a homogeneous solution $-2\nabla_{(a}\xi_{b)}$ to the same equation, thus h^{R}_{ab} can determined only up to a gauge transformation. Even a distribution-valued gauge transformation might be allowed allowed because Appendix F shows that $E_{ab}(\nabla\xi) = 0$, in a distributional sense, even in that extreme case. Thus it is expected that h^{R}_{ab} calculated from (97) might have a non-differentiable part resulting from a singular gauge difference between h^{act}_{ab} and h^{S}_{ab}.

My personal perspective on this situation is reassuring, at least to me, but certainly not rigorous. I have considered about a half-dozen different gravitational self-force problems involving a small point mass orbiting a much larger black hole. In each problem the goal is the calculation of an interesting, well-defined gauge invariant quantity. For each of these, the natural formulation of the problem shows that there are ways to define and to calculate the relevant quantities which are not deterred by a difference in gauge between h^{S}_{ab} and h^{act}_{ab}, even if the difference involves a distribution-valued gauge vector. It appears as though a particularly odious gauge choice might exist for a specific problem, which might interfere with a calculation. However, none of a wide variety of natural choices for a gauge have this difficulty for the problems that I have examined. Specifically, the example in this section appears to avoid any difficult gauge issue.

10.3. Geodesics of the perturbed Schwarzschild metric

A particle of mass μ in a circular orbit about a black hole perturbs the Schwarzschild metric by $h^{\mathrm{act}}_{ab} = \mathrm{O}(\mu)$. The circumstances dictate boundary conditions with no gravitational radiation incoming from infinity or outgoing from the event horizon.

The dynamical timescale for a close orbit is $\mathrm{O}(M)$ and much shorter than the timescale $\mathrm{O}(M^2/\mu)$ for radiation reaction to have a significant effect upon the orbit. Thus the particle will orbit many times before its orbital frequency changes appreciably. Under these conditions, the perturbed metric appears unchanging in a coordinate system that rotates with the particle. Thus, for times much less than the radiation reaction time, there is a Killing vector k^a,

$$\pounds_k(g^0_{ab} + h^{\mathrm{act}}_{ab}) = 0, \tag{99}$$

whose components in the usual Schwarzschild coordinates are

$$k^a\frac{\partial}{\partial x^a} = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}. \tag{100}$$

Let an observer on the particle be equipped with a flashlight which he holds pointing in the plane of the orbit at a fixed orientation with respect to the tangent to the orbit. In other words the orientation of the flashlight is Lie derived by k^a and the beam of light sweeps around the equatorial plane once every orbit. A distant observer, in the equatorial plane, measures the time ΔT between the arrival of two flashes of light from the particle and concludes that $\Omega = 2\pi/\Delta T$ for the particle when the light was emitted. This operational measure of Ω is independent of any gauge choice made for $h^{\rm act}_{ab}$.

The components of the Killing vector k^a in (100) are actually only correct in a particular gauge for which both $\pounds_k g^0_{ab} = 0$ and $\pounds_k h^{\rm act}_{ab} = 0$, individually. Under a gauge transformation the coordinate description of k^a changes by $\mathrm{O}(\mu)$ with $\Delta k^a = -\pounds_\xi k^a$. A choice for ξ^a for which $\pounds_\xi k^a$ is not zero is allowed but it would be very inconvenient and would result in a gauge for which both $\pounds_k g^0_{ab} = \mathrm{O}(h)$ and $\pounds_k h^{\rm act}_{ab} = \mathrm{O}(h)$, even though (99) would still hold. In principle, the geodesics of the light rays could be computed from the particle out to the distant observer in this inconvenient gauge and the orbital period could still be determined. But in practice, this task would be horrendous.

In the convenient gauge, with $\pounds_k h^{\rm act}_{ab} = 0$, the calculation is much easier. From the symmetry, it is clear that the change in Schwarzschild coordinate time between the reception of two light flashes at the observer is the same as the change in Schwarzschild coordinate time at the emission of these flashes. Thus, ΔT measured operationally by a distant observer is equal to the ΔT at the particle for one complete orbit, as long as a gauge is used which respects the inherent symmetry of the example.

We next derive an expression for Ω in (109) which is explicitly gauge invariant for any transformation which respects the symmetry of the example. This includes the possibility of a singular gauge transformation of the type that would transform $h^{\rm act}_{ab}$ from the Regge-Wheeler gauge to the Lorenz gauge.

We let the particle μ move along a geodesic of the perturbed Schwarzschild geometry, $g_{ab} + h_{ab}$, where h_{ab} is the regular remainder $h^{\rm R}_{ab}$ for μ in a circular orbit in the equatorial plane. The geodesic equation for the four-velocity of μ is

$$\frac{\mathrm{d}u_a}{\mathrm{d}s} = \frac{1}{2} u^b u^c \frac{\partial}{\partial x^a}(g_{bc} + h_{bc}) \tag{101}$$

The perturbation breaks the symmetries of the Schwarzschild geometry, and there is no naturally defined energy or angular momentum for the particle. However we let $R(s)$ be the value of r for the particle, and we define specific components of u_a by

$$u_t = -E, \qquad u_\phi = J \qquad u^r = \dot{R}, \qquad \text{and} \qquad u^\theta = 0 \tag{102}$$

where $\dot{}$ denotes a derivative with respect to s. E and J are reminiscent of the particle's energy and angular momentum per unit rest mass.

The components of the geodesic equation (101) are

$$\frac{\mathrm{d}E}{\mathrm{d}s} = -\frac{1}{2} u^a u^b \frac{\partial h_{ab}}{\partial t} \tag{103}$$

$$\frac{\mathrm{d}J}{\mathrm{d}s} = \frac{1}{2} u^a u^b \frac{\partial h_{ab}}{\partial \phi} \tag{104}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{R\dot{R}}{R-2M} + u^a h_{ar}\Big) = \frac{1}{2} u^a u^b \frac{\partial}{\partial r}(g_{ab} + h_{ab}). \tag{105}$$

We are interested in the case when the orbit is nearly circular with $\dot{R}$ only resulting from the effects of energy and angular momentum loss. In this case, $\dot{E}$ and $\dot{J}$ are $\mathrm{O}(h)$, and we look for the additional condition that $\dot{R}$ is also $\mathrm{O}(h)$ to describe the slow inspiral of μ. All of the following equations in this section are assumed to be correct through $\mathrm{O}(h)$, unless otherwise noted.

The normalization of u^a is a first integral of the geodesic equation, and with the assumption that $\dot{R} = \mathrm{O}(h)$ this is

$$-u^a u^b (g_{ab} + h_{ab}) = 1 = \frac{E^2}{1 - 2M/R} - \frac{J^2}{R^2} + u^a u^b h_{ab}. \tag{106}$$

While neither $\partial/\partial t$ nor $\partial/\partial\phi$ is a Killing vector of $g_{ab} + h_{ab}$, the combination, $k^a = \partial/\partial t + \Omega\partial/\partial\phi$ is a Killing vector in a preferred gauge, and u^a is tangent to a trajectory of this Killing vector, up to $\mathrm{O}(h)$. Thus, at a circular orbit $u^a\partial_a h_{bc} = \mathrm{O}(h^2)$ in Schwarzschild coordinates.

A description of the quasi-circular orbits is obtained from (105) and (106) by setting $\ddot{R}$ to zero. The results are (*n.b.* These next two equations were incorrect as written in the original, published version of this manuscript.)

$$E^2 = \frac{(R-2M)^2}{R(R-3M)}[1 - u^a u^b h_{ab} - \frac{1}{2} R\partial_r(u^a u^b h_{ab})] \tag{107}$$

and

$$J^2 = \frac{MR^2}{R-3M}(1 - u^a u^b h_{ab}) - \frac{R^3(R-2M)}{2(R-3M)}\partial_r(u^a u^b h_{ab}). \tag{108}$$

Also the angular velocity, Ω, of a circular orbit as measured at infinity is

$$\Omega^2 = (\mathrm{d}\phi/\mathrm{d}t)^2 = (u^\phi/u^t)^2 = M/R^3 - \frac{R-3M}{2R^2} u^a u^b \partial_r h_{ab}. \tag{109}$$

Finally,

$$(E - \Omega J)^2 = (1 - 3M/R)(1 - u^a u^b h_{ab} + R u^a u^b \partial_r h_{ab}/2). \tag{110}$$

These equations give E, J, Ω and $E - \Omega J$ for a circular orbit in terms of the radius of the orbit R and the metric perturbation h_{ab}. We can consider the effect on these expressions of a gauge transformation which preserves the $\partial/\partial t + \Omega\partial/\partial\phi$ symmetry of the problem. The analysis uses descriptions of gauge transformations found, for example, in references [15] and [17]. Here we present only the results.

The orbital frequency Ω and $E - \Omega J$ are both invariant under a gauge transformation, while E and J are not. However, both $\mathrm{d}E/\mathrm{d}s$ and $\mathrm{d}J/\mathrm{d}s$ in (103) and (104) are gauge invariant. This latter result might have been anticipated by using an operational definition of energy and angular momentum loss as measured by a distant observer, and by finding a relationship which joins the right hand sides of (103) and (104) with the matching conditions at the particle for the differential equations which describe the metric perturbation. This relationship is straightforward but quite tedious to demonstrate directly.

The energy and the angular momentum measured by a distant observer are gauge invariant. At zeroth order in the perturbation the energy is just M, the mass of the black hole. The $\ell = 0$ metric perturbation gives the $\mathrm{O}(\mu)$ contribution to the energy and this is just μE, which is an $\mathrm{O}(\mu)$ quantity independent of gauge but not relying upon the $\mathrm{O}(\mu)$ terms in (107). The contribution of the gravitational self-force to the energy measured at a large distance shows up only at second order in μ; to calculate this effect requires going to second order perturbation analysis. Similar statements hold for the angular momentum measured at a large distance and J.

For a circular orbit the radius, R, and both E and J all depend upon the choice of gauge for h_{ab}. However, Ω is defined in terms of a measurement made at infinity, and $E - \Omega J$ is the contraction of u_a with the Killing vector ξ^a; hence, these latter two quantities are independent of the gauge, and this has been demonstrated explicitly allowing for distribution valued gauge transformations.

The gauge invariance of Ω has an interesting twist. While Ω is gauge invariant, the Schwarzschild radius of the orbit is not. A typical gauge vector ξ^a has a radial component which changes the coordinate description R of the orbit. This affects Ω through the M/R^3 term in (109). This radial component of ξ^a also changes the $u^a u^b \partial_r h_{ab}$ in a manner that leaves the right hand side of (109) unchanged. Equation (109) gives the same result whether evaluated in a limit from outside the orbit or inside, in the event that h_{ab} is not differentiable at the orbit; this result follows from analysis of the jump conditions on h_{ab} at the orbit.

11. Future prospects

Within the next year or two important applications of gravitational self-force analyses will be viable.

For some time, it has been possible to calculate energy and angular momentum loss by a small mass in an equatorial orbit about a Kerr black hole using the Teukolsky [43] formalism, which involves the Newman-Penrose [44] scalars ψ_0 and ψ_4. For orbits off the equatorial plane this is not good enough. For gravitational waveform prediction, it is also necessary to calculate the dissipative change in the third "constant" of the motion, the Carter constant C, due to gravitational radiation. Energy and angular momentum loss may be determined by finding the flux at a large distance in a gauge invariant manner. There is no corresponding "Carter constant flux." However, Lousto and Whiting [45, 46] describe progress in determining metric perturbations from ψ_0 or ψ_4. And Mino [34] has proposed a method for determining $\mathrm{d}C/\mathrm{d}t$ which depends upon these metric perturbations.

Self-force calculations in the Schwarzschild geometry are much easier, and progress is likely to be rapid both in connecting results with post-Newtonian analyses and in tracking the phase of gravitational radiation in an extreme mass-ratio binary.

11.1. Evolution of the phase during quasi-circular inspiral

One application of self-force analysis is to track of the phase of the gravitational wave from a small object while it orbits a large black hole many times. For this task, Burko [47] has emphasized the necessity of using higher order perturbation theory to calculate properly the effects of the conservative part of the self-force. Here, we follow his lead, and estimate the number of orbits which can be tracked by use of analysis with different levels of sophistication.

For definiteness, we assume that a small mass μ is undergoing slow, quasi-stationary inspiral about a Schwarzschild black hole of mass M and that the orbit is relativistic so that M gives the dynamical time scale. A gauge-invariant E of the orbiting particle is defined in terms of the mass as measured at infinity,

$$\mu E \equiv M_\infty - M. \tag{111}$$

If we know $\Omega(E)$ and also $\mathrm{d}E/\mathrm{d}t$, then the assumption of quasi-circular inspiral provides

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{\mathrm{d}\Omega}{\mathrm{d}E}\frac{\mathrm{d}E}{\mathrm{d}t}. \tag{112}$$

Let Ω_o be the orbital frequency at $t = 0$. The phase of the orbit is then

$$\begin{aligned}\phi(t) &= \int_0^t \Omega(E(t))\,\mathrm{d}t \qquad (113)\\ &= \int_0^t \left(\Omega_o + t\left[\frac{\mathrm{d}\Omega}{\mathrm{d}E}\frac{\mathrm{d}E}{\mathrm{d}t}\right] + \cdots\right)\mathrm{d}t\end{aligned}$$

after a Taylor expansion.

At the lowest level of approximation Ω and E are given by the geodesic equation in the Schwarzschild metric. The solution of the first-order metric monopole perturbation problem, via Regge-Wheeler [15] analysis, gives

$$E \approx E_{1\mathrm{st}} \equiv -u_t(\text{circular orbit}) \qquad (114)$$

First order analysis, of the sort that was available in the 1970's, also allows for the determination of $(dE/dt)_{1\mathrm{st}}$. To obtain new information [47] regarding E, Ω and $\mathrm{d}E/\mathrm{d}t$, requires second-order perturbation analysis, which presupposes the solution of the first-order self-force problem. Second and higher order analysis would provide

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \left[\frac{\mathrm{d}\Omega}{\mathrm{d}E}\frac{\mathrm{d}E}{\mathrm{d}t}\right]_{1\mathrm{st}}[1 + \Delta_{2\mathrm{nd}} + \cdots], \qquad (115)$$

where $\Delta_{2\mathrm{nd}} = \mathrm{O}(\mu/\mathcal{R})$. With second or higher order analysis, the phase is

$$\begin{aligned}\phi &= \int_0^t \left(\Omega_o + t\left[\frac{d\Omega}{dE}\frac{dE}{dt}\right]_{1\mathrm{st}}[1 + \Delta_{2\mathrm{nd}} + \cdots]\right)dt\\ &= t\Omega_o + \frac{1}{2}t^2\left[\frac{d\Omega}{dE}\frac{dE}{dt}\right]_{1\mathrm{st}}[1 + \Delta_{2\mathrm{nd}} + \cdots] \qquad (116)\end{aligned}$$

after integration.

Consider the size of the contribution to the phase of the different terms of

$$\frac{1}{2}t^2\left[\frac{d\Omega}{dE}\frac{dE}{dt}\right]_{1\mathrm{st}}[1 + \Delta_{2\mathrm{nd}} + \cdots] \approx \frac{1}{2}t^2\left[\mathrm{O}\left(\frac{\mu}{M^3}\right)\right]_{1\mathrm{st}}\left(1 + \left[\mathrm{O}\left(\frac{\mu}{M}\right)\right]_{2\mathrm{nd}} + \cdots\right). \qquad (117)$$

If radiation reaction is not included in the analysis, then none of this term, of order $\frac{1}{2}t^2\mu/M^3$, is included. This would lead to a phase error of one full cycle after a time of order $t_{\mathrm{dp}} = M\sqrt{M/\mu}$, which is known as the de-phasing timescale.

If only the first-order radiation reaction term is included, then the $\frac{1}{2}t^2\mu^2/M^4$ term is not included and leads to a phase error of one full cycle after a time of order $t_{\mathrm{rr}} = M^2/\mu$. This is the radiation reaction timescale.

If second-order radiation reaction is also included, then the $\cdots$ terms of order $\frac{1}{2}t^2\mu^3/M^5$ are not included and create a phase error of one full cycle after a time of order $t_{2\mathrm{nd}} = (M^2/\mu)\sqrt{M/\mu})$. This is the second-order timescale.

These same results are restated by noting that geodesic motion loses one cycle of phase information after order $\sqrt{M/\mu}$ orbits. First order perturbation theory loses one cycle of phase information after order M/μ orbits. And second order perturbation theory loses one cycle of phase information after order $(M^2/\mu)\sqrt{M/\mu}$ orbits.

These estimates describe the difficulty involved in tracking the phase of an orbit over an increasing number of orbits.

11.2. Connection with post-Newtonian analyses

An effort is now underway to find the effects of the gravitational self-force on a number of parameters related to orbits in the Schwarzschild metric. The first interesting results will be the orbital frequency Ω and the rate of precession of the perihelion for a slightly eccentric orbit. Other parameters which can be calculated with self-force analysis for circular orbits are $E - \Omega J$ and a gauge invariant measure of the radius of the orbit (see section 9.2).

First order perturbation theory coupled with self-force analysis will provide the $\mathrm{O}(\mu/M)$ effect on the innermost stable circular orbit (ISCO), as well as the effect on the angular frequency of the ISCO. Currently, there is no firm prediction as to whether the self-force moves the ISCO in or out. Some recent scalar-field self-force results [48] show that the ISCO moves in and the frequency of the ISCO increases; but there is no clear generalization of this result to gravitation.

More interesting quantities will be available with second order perturbation calculations, which now appear feasible. These include the energy, angular momentum, and the rate of radiative loss of these quantities. Eventually, second order gravitational wave-forms will be calculated.

One early goal of self-force analysis is to make contact with post-Newtonian results. To do so requires that the quantities being calculated via perturbation analysis match up precisely with those from post-Newtonian analysis.

Post-Newtonian analyses are most reliable with slow speeds and weak gravitational fields, and they easily accommodate comparable masses in a binary. Perturbation analyses are most reliable with an extreme mass ratio, but they accommodate fast speeds and strong fields. For, say, a $3M_\odot$ black hole orbiting a $20M_\odot$ black hole near its innermost orbit, the mass ratio is not very extreme, the speeds are not very slow and the fields are not very weak. Nevertheless, for this situation both post-Newtonian and perturbation methods will be able to estimate properties of the system. A comparison of these estimates will certainly highlight the strong and weak aspects of each approach.

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Appendix A. Perturbed Bianchi identity

The Bianchi identity is

$$\nabla_c R_{dea}{}^b + \nabla_e R_{cda}{}^b + \nabla_d R_{eca}{}^b = 0. \tag{A.1}$$

Contraction on c and b implies that

$$\nabla_b R_{dea}{}^b = 0 \tag{A.2}$$

for a vacuum solution of the Einstein equations. This result is used often in the derivations of identities involving $E_{ab}(h)$.

The definition of the operator E_{ab} for a vacuum spacetime is

$$2E_{ab}(h) = \nabla^2 h_{ab} + \nabla_a \nabla_b h - 2\nabla_{(a}\nabla^c h_{b)c} + 2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c\nabla^d h_{cd} - \nabla^2 h), \quad \text{(A.3)}$$

so that

$$\begin{aligned} 2\nabla^a E_{ab}(h) &= \nabla^a\nabla^c\nabla_c h_{ab} + \nabla^a\nabla_a\nabla_b h - \nabla^a\nabla_a\nabla^c h_{bc} - \nabla^a\nabla_b\nabla^c h_{ac} \\ &\quad + 2(\nabla^a R_a{}^c{}_b{}^d)h_{cd} + 2R^{ac}{}_b{}^d\nabla_a h_{cd} + \nabla_b\nabla^c\nabla^d h_{cd} - \nabla_b\nabla^c\nabla_c h. \\ &= \nabla^a\nabla^c\nabla_a h_{cb} - \nabla^a\nabla_a\nabla^c h_{bc} - \nabla^a\nabla_b\nabla^c h_{ac} + \nabla_b\nabla^a\nabla^c h_{ac} \\ &\quad + R^{ac}{}_b{}^d\nabla_c h_{ad} + 2R^{ac}{}_b{}^d\nabla_a h_{cd} \\ &= 0. \end{aligned} \quad \text{(A.4)}$$

The second equality follows after use of the Ricci identity to interchange the order of derivatives on the first, second and last terms as well as repeated uses of $R_{ab} = 0$ and (A.2) for vacuum spacetimes. The final result follows after use of the Ricci identity on the first two terms and on the second two terms of the second equality, and the application of symmetries of the Riemann tensor on the remainder.

If h_{ab} is not C^3 then $\nabla^a E_{ab}(h) = 0$ in a distributional sense. To show this, choose an arbitrary, smooth test vector field λ^a with compact support. Consider the integral of $\lambda^b\nabla^a E_{ab}(h)$ over a sufficiently large region. Integrate by parts once and discard the surface term. Next use (E.4) and discard the surface terms to obtain an integral of $h^{ab}E_{ab}(\nabla\lambda)$. This integral is zero from (D.1). These steps also provide an alternative derivation of (A.4) in the event that h_{ab} is, in fact, C^3.

Appendix B. Formal nth order perturbation analysis

In general perturbation analysis, let the g_{ab} of (15) be an exact solution to the vacuum Einstein equations, g^0_{ab}, and iteratively define

$$g^{(n)}_{ab} = g^{(n-1)}_{ab} + h^{(n)}_{ab} \quad \text{(B.1)}$$

where

$$h^{(n)}_{ab} = \mathrm{O}(\mu^n). \quad \text{(B.2)}$$

Assume that we are given $g^{(n-1)}_{ab}$ and $T^{(n)}_{ab}$ with

$$G^{(n-1)}_{ab} - 8\pi T^{(n)}_{ab} = \mathrm{O}(\mu^n). \quad \text{(B.3)}$$

If $h^{(n)}_{ab}$ is a solution of

$$E_{ab}(h^{(n)}) = G^{(n-1)}_{ab} - 8\pi T^{(n)}_{ab} + \mathrm{O}(\mu^{n+1}), \quad \text{(B.4)}$$

then it follows from the definition of the operator $E_{ab}(h)$ in (14) that

$$G^{(n)}_{ab} - 8\pi T^{(n)}_{ab} = \mathrm{O}(\mu^{n+1}), \quad \text{(B.5)}$$

and $h^{(n)}_{ab}$ is an $\mathrm{O}(\mu^n)$ improvement to the approximate solution to the Einstein equations.

The Bianchi identity implies that

$$\nabla^a E_{ab}(h) = 0 \quad \text{(B.6)}$$

for any symmetric C^3 tensor field h_{ab}, as shown in Appendix A. It is also shown that if h_{ab} is not C^3 then (B.6) holds in a distributional sense. Thus an integrability condition of (B.4) is that

$$\nabla^a\big(G^{(n-1)}_{ab} - 8\pi T^{(n)}_{ab}\big) = \mathrm{O}(\mu^{n+1}). \tag{B.7}$$

Note, however, that

$$\begin{aligned}\nabla^a\big(G^{(n-1)}_{ab} - 8\pi T^{(n)}_{ab}\big) &= \nabla^a_{(n-1)}\big(G^{(n-1)}_{ab} - 8\pi T^{(n)}_{ab}\big)\\ &\quad + \Gamma^a_{ac}\big(G^{(n-1)c}_{b} - 8\pi T^{(n)c}_{b}\big) - \Gamma^c_{ab}\big(G^{(n-1)a}_{c} - 8\pi T^{(n)a}_{c}\big),\end{aligned} \tag{B.8}$$

where $\nabla^a_{(n-1)}$ is the derivative operator of $g^{(n-1)}_{ab}$, and Γ^a_{bc} is the connection relating the derivative operators ∇^a and $\nabla^a_{(n-1)}$. The Bianchi identity implies that

$$\nabla^a_{(n-1)} G^{(n-1)}_{ab} = 0, \tag{B.9}$$

and the terms in (B.8) involving Γ^a_{bc} are order μ^{n+1} because of (B.3) and the fact that $\Gamma^a_{bc} = \mathrm{O}(\mu)$. Thus, the approximate vanishing of the right hand side of (B.8) is the integrability condition for (B.4),

$$\nabla^a_{(n-1)} T^{(n)}_{ab} = \mathrm{O}(\mu^{n+1}). \tag{B.10}$$

In other words, before (B.4) can be solved for $h^{(n)}_{ab}$, it is necessary that the perturbing stress tensor be adjusted to be conserved with the metric $g^{(n-1)}_{ab}$ and to satisfy (B.10).

Appendix C. $\nabla_b T^{ab} = 0$ implies the geodesic equation for a point mass

We follow an example in reference [49]. In (19), $\delta^4(x^a - X^a(s))/\sqrt{-g}$ is a scalar field, and the factor u^b may be defined as a vector field by extension, in any smooth manner, away from the world line. Then,

$$\begin{aligned}(g^c{}_a + u^c u_a)\nabla_b T^{ab} &= \mu (g^c{}_a + u^c u_a)\int_{-\infty}^{\infty}\Bigg[\frac{(\nabla_b u^a)u^b}{\sqrt{-g}}\delta^4(x^a - X^a(s))\\ &\qquad + u^a\nabla_b\left(\frac{u^b}{\sqrt{-g}}\delta^4(x^a - X^a(s))\right)\Bigg]\,\mathrm{d}s\\ &= \mu\int_{-\infty}^{\infty}\frac{(\nabla_b u^a)u^b}{\sqrt{-g}}\delta^4(x^a - X^a(s))\,\mathrm{d}s\end{aligned} \tag{C.1}$$

where the integrals are zero away from the world line and the second equality follows from properties of the projection operator $g^c{}_a + u^c u_a$. If $\nabla_b T^{ab} = 0$, then it necessarily follows that the coefficient of the delta function must be zero for all proper times. A consequence is that $u^b\nabla_b u^a = 0$, the geodesic equation.

A more formal proof of this result is in Poisson's review of the self-force [13], Section 5.3.1.

Appendix D. Gauge invariance of $E_{ab}(h)$

For a background geometry which is a vacuum solution of the Einstein equations, an infinitesimal gauge transformation, $x^a_{\mathrm{new}} = x^a + \xi^a$, with $\xi^a = \mathrm{O}(\mu)$ changes the metric perturbation, $h^{\rm new}_{ab} = h_{ab} - 2\nabla_{(a}\xi_{b)} + \mathrm{O}(\mu^2)$. But the operator $E_{ab}(h)$ is invariant under such a coordinate transformation,

$$E_{ab}(\nabla\xi) = 0. \tag{D.1}$$

This result follows immediately from the fact that the change in the perturbation of the Einstein tensor E_{ab} under a gauge transformation is the Lie derivative of the background Einstein tensor $\pounds_\xi G_{ab}$. For a vacuum background spacetime, this is zero.

Equation (D.1) also follows from direct substitution into

$$2E_{ab}(h) = \nabla^2 h_{ab} + \nabla_a\nabla_b h - 2\nabla_{(a}\nabla^c h_{b)c} + 2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c\nabla^d h_{cd} - \nabla^2 h) \tag{D.2}$$

with $h_{ab} = 2\nabla_{(a}\xi_{b)}$. It is easiest to consider the factor of g_{ab} separately,

$$\begin{aligned}\text{factor of } g_{ab} &= \nabla^c\nabla^d\nabla_c\xi_d + \nabla^c\nabla^d\nabla_d\xi_c - 2\nabla^a\nabla_a\nabla^b\xi_b \\ &= 2\nabla^c\nabla^d\nabla_c\xi_d - 2\nabla^a\nabla_a\nabla^b\xi_b \\ &= 0.\end{aligned} \tag{D.3}$$

The second equality follows after use of the Ricci identity on the first two indices of the second term, use of $R_{ab} = 0$ for a vacuum spacetime and a relabeling of the indices. The final result follows after use of the Ricci identity on the second term of the second equality and use of $R_{ab} = 0$ for a vacuum spacetime. With $h_{ab} = 2\nabla_{(a}\xi_{b)}$, the remainder of $E_{ab}(2\nabla\xi)$ is

$$\begin{aligned}\text{remainder} = {} & \nabla^c\nabla_c\nabla_a\xi_b + \nabla^c\nabla_c\nabla_b\xi_a + 2\nabla_a\nabla_b\nabla^c\xi_c \\ & - \nabla_a\nabla^c\nabla_b\xi_c - \nabla_a\nabla^c\nabla_c\xi_b - \nabla_b\nabla^c\nabla_a\xi_c - \nabla_b\nabla^c\nabla_c\xi_a \\ & + 2R_a{}^c{}_b{}^d\nabla_c\xi_d + 2R_a{}^c{}_b{}^d\nabla_d\xi_c.\end{aligned} \tag{D.4}$$

The analysis of this expression is lengthy but not difficult. It begins with using the Ricci identity upon the second and third indices of the first, second, fourth and sixth terms and upon the first and second indices of the fifth and seventh terms. The resulting terms with three derivatives may be paired up in a way to use the Ricci identity again and to reduce the entire expression to one involving only single derivatives. This also requires application of (A.2). That the entire expression is zero, then follows from the symmetries of the Riemann tensor.

Appendix E. Green's theorem for E_{ab}

The operator $E_{ab}(h)$ in (15), with an arbitrary tensor k^{ab}, satisfies the identity

$$2k^{ab}E_{ab}(h) = \nabla_c F^c(k,h) - \langle k^{ab}, h_{ab}\rangle, \tag{E.1}$$

where

$$F^c(k,h) \equiv k^{ab}\nabla^c h_{ab} - \frac{1}{2}k\nabla^c h - 2(k^{cb} - \frac{1}{2}g^{cb}k)\nabla^a(h_{ab} - \frac{1}{2}g_{ab}h) \tag{E.2}$$

and

$$\begin{aligned}\langle k^{ab}, h_{ab}\rangle \equiv {} & \nabla^c k^{ab}\nabla_c h_{ab} - \frac{1}{2}\nabla^c k\nabla_c h \\ & - 2\nabla_a(k^{ac} - \frac{1}{2}g^{ac}k)\nabla^b(h_{bc} - \frac{1}{2}g_{bc}h) - 2k^{ab}R_a{}^c{}_b{}^d h_{cd}.\end{aligned} \tag{E.3}$$

Note that the "inner product," $\langle k^{ab}, h_{ab}\rangle = \langle h^{ab}, k_{ab}\rangle$ is symmetric under the interchange of h^{ab} and k_{ab}. It follows that

$$k^{ab}E_{ab}(h) - h^{ab}E_{ab}(k) = \frac{1}{2}\nabla_c\left[F^c(k,h) - F^c(h,k)\right]. \tag{E.4}$$

Which is a tensor version of Green's theorem for the differential operator $E_{ab}(h)$.

The derivation of equation (E.1) is straightforward. Contract (15) with an arbitrary symmetric tensor k^{ab}, and move k^{ab} inside ∇_a in each term by "differentiating by parts." the divergence terms determine $F^c(k,h)$.

Appendix F. Singular gauge transformations

Let ξ^a be a, possibly distribution valued, vector field. And let $h_{ab} = -2\nabla_{(a}\xi_{b)}$, as for a gauge transformation. Also, let k_{ab} be a smooth "test" tensor with compact support. We use the operator $E_{ab}(h)$ in (15), then

$$\begin{aligned}\int k^{ab}E_{ab}(h)\sqrt{-g}\,\mathrm{d}^4x &= \int h^{ab}E_{ab}(k)\sqrt{-g}\,\mathrm{d}^4x \\ &= -2\int(\nabla^a\xi^b)E_{ab}(k)\sqrt{-g}\,\mathrm{d}^4x, \qquad \text{(F.1)}\end{aligned}$$

from (E.4), after dropping the divergence term. An integration by parts and application of the perturbed Bianchi identity (A.4) yields

$$\begin{aligned}\int k^{ab}E_{ab}(h)\sqrt{-g}\,\mathrm{d}^4x &= 2\int \xi^b\nabla^a\left[E_{ab}(k)\right]\sqrt{-g}\,\mathrm{d}^4x \\ &= 0. \qquad \text{(F.2)}\end{aligned}$$

Thus, we demonstrate that given a solution to the inhomogeneous, perturbed Einstein equations (13), even a distributional gauge transformation leaves a distributional valued metric perturbation that continues to satisfy the perturbed Einstein equations in this distributional sense.

Appendix G. Black hole moving through an external background geometry

When a small Schwarzschild black hole of mass m moves through a background spacetime, the hole's metric is perturbed by quadrupole tidal forces arising from ${}_2H_{ab}$ in (24) or (53), and the actual metric near the black hole, including the quadrupole perturbation, is

$$g^{\mathrm{act}}_{ab} = g^{\mathrm{Schw}}_{ab} + {}_2h_{ab} + \mathrm{O}(r^4/\mathcal{R}^4), \qquad \text{(G.1)}$$

where the quadrupole metric perturbation ${}_2h_{ab}$ is a solution of

$$E^{\mathrm{Schw}}_{ab}({}_2h) = 0. \qquad \text{(G.2)}$$

Here E^{Schw}_{ab} is the Schwarzschild geometry version of the operator given in (15). The appropriate boundary conditions for (G.2) are that the perturbation be well behaved on the event horizon and that ${}_2h_{ab} \to {}_2H_{ab}$ in the buffer region, where $\mu \ll r \ll \mathcal{R}$.

Our analyses of the boundary conditions and solutions of equation (G.2) for slow motion are very strongly influenced by Poisson's recent analysis [27, 28, 29] of the same situation. In this appendix we describe ${}_2h_{ab}$ up to a remainder of $\mathrm{O}(r^4/\mathcal{R}^4)$.

The appropriate boundary conditions at the future event horizon are that h_{ab} be finite and well behaved in a coordinate system which is well behaved itself. The Eddington-Finkelstein ingoing coordinates are satisfactory, and

$$V = t + r_* \quad \text{and} \quad R = r, \qquad \text{(G.3)}$$

where $r_* = r + 2m\ln(r/2m - 1)$; the angles θ and ϕ remain unchanged.

The odd and even parity parts of the metric perturbation are governed by the Regge-Wheeler [15] and Zerilli [17] equations, respectively. For our purposes, we change these equations from the Schwarzschild t,r to the Eddington-Finkelstein V, R. For example, the Regge-Wheeler equation for $W(V,R)$ becomes

$$2\frac{\partial^2 W}{\partial V \partial R} + \left(1 - \frac{2m}{R}\right)\frac{\partial^2 W}{\partial R^2} + \frac{2m}{R^2}\frac{\partial W}{\partial R} - 6(R-m)\frac{W}{R^3} = 0, \tag{G.4}$$

where we have assumed that the angular dependence of W corresponds to a linear combination of $\ell = 2$ spherical harmonics. We next assume that W is slowly changing in V and accordingly let

$$W(V,R) = \mathcal{B}W_0(R) + \mathcal{B}'W_1(R) + \ldots \quad \text{where} \quad \mathcal{B}' = d\mathcal{B}(V)/dV. \tag{G.5}$$

$\mathcal{B}$ is a function of V, and of the angles, that is related ultimately to the time-dependent external quadrupole moment of the geometry through which the black hole is moving. Thus $\mathcal{B} = \mathrm{O}(\mathcal{R}^{-2})$ and $\mathcal{B}' = \mathrm{O}(\mathcal{R}^{-3})$, in keeping with the requirement of slow time dependence.

With the form (G.5) substituted into (G.4), we separate the terms of $\mathrm{O}(\mathcal{R}^{-2})$ from those of $\mathrm{O}(\mathcal{R}^{-3})$ to obtain two equations. One is an ordinary, homogeneous differential equation for $W_0(R)$, and the second is for $W_1(R)$ with a source term from the $\partial^2(\mathcal{B}W_0)/\partial V \partial R$ term. An analytic solution of the first equation is $W_0(R) = R^3$. The solution of the second for $W_1(R)$ is also analytic but more complicated, and constants of integration may be chosen so that W_1 is well behaved at the horizon. This procedure thus provides a general solution for $W(V,R)$, up to a remainder of $\mathrm{O}(\mathcal{R}^{-4})$. The even parity Zerilli equation may be solved in a similar manner or by using the simple relationship between solutions of the Regge-Wheeler and Zerilli equations [50].

From the solutions of the Regge-Wheeler and Zerilli equations, the actual metric perturbations are determined by taking derivatives of the master variables. This results in a metric perturbation, in the Regge-Wheeler gauge, whose non-zero Schwarzschild components, as functions of V and r, are

$$\begin{aligned} h_{tt} = &-(r-2m)^2[\mathcal{E}^{(2)} - 2m\ln(r/2m)\,\mathcal{E}'^{(2)}] \\ &+ \frac{1}{3r^2}(3r^5 - 12r^4m + 36m^3r^2 - 16m^4r - 8m^5)\mathcal{E}'^{(2)} \end{aligned} \tag{G.6}$$

$$h_{tr} = -\frac{r(2r^3 - 3mr^2 - 6m^2r + 6m^3)}{3(r-2m)}\mathcal{E}'^{(2)} \tag{G.7}$$

$$\begin{aligned} \frac{1}{2}h^{\mathrm{trc}} = &-(r^2 - 2m^2)[\mathcal{E}^{(2)} - 2m\ln(r/2m)\,\mathcal{E}'^{(2)}] \\ &+ \frac{1}{3r}(3r^4 - 18m^2r^2 - 12m^3r + 8m^4)\mathcal{E}'^{(2)} \end{aligned} \tag{G.8}$$

$$\begin{aligned} h_{rr} = &-r^2[\mathcal{E}^{(2)} - 2m\ln(r/2m)\,\mathcal{E}'^{(2)}] \\ &+ \frac{1}{3}\frac{(3r^5 - 12r^4m + 36m^3r^2 - 16m^4r - 8m^5)}{(r-2m)^2}\mathcal{E}'^{(2)} \end{aligned} \tag{G.9}$$

$$\begin{aligned} h^{\mathrm{od}}_{tA} = &\frac{1}{3}r(r-2m)[\mathcal{B}^{(2)}_A - 2m\ln(r/2m)\,\mathcal{B}'^{(2)}_A] \\ &- \frac{1}{9r^2}(3r^5 - 6r^4m - 12r^3m^2 + 12r^2m^3 + 8rm^4 + 8m^5)\mathcal{B}'^{(2)}_A \end{aligned} \tag{G.10}$$

$$h^{\rm od}_{rA} = \frac{r^4}{12(r-2m)}\mathcal{B}'^{(2)}_A. \tag{G.11}$$

This metric perturbation was first derived by Poisson [29] in a different gauge.

In these expressions $\mathcal{B}^{(2)}$ and $\mathcal{E}^{(2)}$ are V-dependent linear combinations of the $\ell = 2$ spherical harmonic functions $Y_{2,m}(\theta,\phi)$. The V and R coordinate components are all well behaved on the future event horizon. The V-dependence of $\mathcal{E}^{(2)}$ and $\mathcal{B}^{(2)}$ shows that the metric perturbation propagates toward the black hole from a great distance as expected.

To make contact with the actual, external geometry it is useful to expand the expressions given in (G.6)-(G.11) for r in the buffer region, where $m \ll r \ll \mathcal{R}$, and we take advantage of the fact that $\mathcal{B}'^{(2)}$ and $\mathcal{E}'^{(2)}$ are $\mathrm{O}(\mathcal{R}^{-3})$. Thus, for $r_* \ll \mathcal{R}$ a Taylor series about $V = t$ provides

$$\begin{aligned}\mathcal{B}(V) &= \mathcal{B}(t + r_*)\\ &= \mathcal{B}(t) + r_*\frac{d\mathcal{B}(t)}{dt} + \mathrm{O}(r_*^2/\mathcal{R}^4).\end{aligned} \tag{G.12}$$

For $m \ll r \ll \mathcal{R}$, the even parity part of the metric perturbation in Schwarzschild coordinates is

$$\begin{aligned}h^{\rm ev}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = &-\mathcal{E}^{(2)}\left[(r-2m)^2\,\mathrm{d}t^2 + r^2\,\mathrm{d}r^2 + (r^2-2m^2)\sigma_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B\right]\\ &+\frac{16m^6}{15r^4}\dot{\mathcal{E}}^{(2)}\left[2(r+m)\,\mathrm{d}t^2 + 2(r+5m)\,\mathrm{d}r^2 + (2r+5m)\sigma_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B\right]\\ &-2\frac{r(2r^3-3mr^2-6m^2r+6m^3)}{3(r-2m)}\dot{\mathcal{E}}^{(2)}\,\mathrm{d}t\,\mathrm{d}r + \mathrm{O}(m^8\dot{\mathcal{E}}^{(2)}/r^5).\end{aligned} \tag{G.13}$$

and the odd parity part is

$$\begin{aligned}h^{\rm od}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = 2&\left[\frac{r}{3}(r-2m)\mathcal{B}^{(2)}_A + \frac{16m^6}{45r^4}(3r+4m)\dot{\mathcal{B}}^{(2)}_A\right]\mathrm{d}t\,\mathrm{d}x^A\\ &+2\frac{r^4}{12(r-2m)}\dot{\mathcal{B}}^{(2)}_A\,\mathrm{d}r\,\mathrm{d}x^A + \mathrm{O}(m^8\dot{\mathcal{B}}^{(2)}/r^5).\end{aligned} \tag{G.14}$$

In this form $\mathcal{E}^{(2)}$ and $\mathcal{B}^{(2)}$ are considered functions of t and $\dot{\mathcal{E}}^{(2)}$ denotes the t derivative of $\mathcal{E}^{(2)}$.

References

[1] L. Barack and A. Ori, Phys. Rev. D **64**, 124003 (2001).
[2] S. Detweiler and E. Poisson, Phys. Rev. D **69**, 084019 (2004).
[3] J. Jackson, *Classical Electrodynamics, Third Edition* (Wiley, New York, 1998).
[4] P. A. M. Dirac, Proc. R. Soc. (London) **A167**, 148 (1938).
[5] B. S. DeWitt and R. W. Brehme, Ann. Phys. (N.Y.) **9**, 220 (1960).
[6] S. Detweiler and B. F. Whiting, Phys. Rev. D **67**, 024025 (2003).
[7] Y. Mino, M. Sasaki, and T. Tanaka, Phys. Rev. D **55**, 3457 (1997).
[8] F. K. Manasse, J. Math. Phys. **4**, 746 (1963).
[9] P. D. D'Eath, Phys. Rev. D **11**, 1387 (1975).
[10] R. Kates, Phys. Rev. D **22**, 1853 (1980).
[11] K. Alvi, Phys. Rev. D **61**, 124013 (2000).
[12] S. Detweiler, Phys. Rev. Lett. **86**, 1931 (2001).
[13] E. Poisson, Living Rev. Relativity **7**, 6 (2004), http://www.livingreviews.org/lrr-2004-6.
[14] T. Quinn and R. Wald, Phys. Rev. D **56**, 3381 (1997).
[15] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[16] M. Davis, R. Ruffini, W. H. Press, and R. Price, Phys. Rev. Lett. **27**, 1466 (1971).
[17] F. J. Zerilli, Phys. Rev. D **2**, 2141 (1970).

[18] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[19] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Fransisco, 1973).
[20] F. K. Manasse and C. W. Misner, J. Math. Phys. **4**, 735 (1963).
[21] W.-Q. Li and W.-T. Ni, J. Math. Phys. **20**, 1473 (1979).
[22] K. S. Thorne and J. B. Hartle, Phys. Rev. D **31**, 1815 (1985).
[23] X.-H. Zhang, Phys. Rev. D **34**, 991 (1986).
[24] S. Detweiler, E. Messaritaki, and B. F. Whiting, Phys. Rev. D **67**, 104016 (2003).
[25] E. Messaritaki, University of Florida Ph.D. Dissertation (2003), http://arxiv.org/abs/gr-qc/0308047.
[26] K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980).
[27] E. Poisson, Phys. Rev. D **69**, 084007 (2004).
[28] E. Poisson, Phys. Rev. D **70**, 084044 (2004).
[29] E. Poisson, (2005), http://arXiv:gr-qc/0501032 v1.
[30] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw Hill, New York, 1978).
[31] R. Sachs, in *Relativity, Groups and Topology*, edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1964).
[32] J. M. Stewart and M. Walker, Proc. R. Soc. (London) **341**, 49 (1974).
[33] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[34] Y. Mino, Phys. Rev. D **67**, 084027 (2003).
[35] L. Barack and A. Ori, Phys. Rev. D **61**, 061502(R) (2000).
[36] L. Barack, Phys. Rev. D **62**, 084027 (2000).
[37] L. Barack and A. Ori, Phys. Rev. D **66**, 084022 (2002).
[38] L. Barack and A. Ori, Phys. Rev. D **67**, 024029 (2003).
[39] Y. Mino, H. Nakano, and M. Sasaki, Prog. Theor. Phys. **108**, 1039 (2002).
[40] L. Barack, Y. Mino, H. Nakano, A. Ori, and M. Sasaki, Phys. Rev. Lett. **88**, 091101 (2002).
[41] C. O. Lousto, Phys. Rev. Lett. **84**, 5251 (2000).
[42] D.-H. Kim, http://arxiv.org/abs/gr-qc/0402014 (unpublished).
[43] S. A. Teukolsky, Astrophys. J. **185**, 635 (1973).
[44] E. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).
[45] C. O. Lousto and B. F. Whiting, Phys. Rev. D **66**, (2002).
[46] C. O. Lousto, (2005), http://arXiv:gr-qc/0501088.
[47] L. M. Burko Phys. Rev. D **67**, 084001, (2003).
[48] L. M. Diaz-Rivera, E. Messaritaki, B. Whiting, and S. Detweiler, Phys. Rev. D **70**, 124018 (2004).
[49] A. P. Lightman, W. H. Press, R. H. Price, and S. A. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton University Press, Princeton, 1979).
[50] S. Chandrasekhar and S. Detweiler, Proc. R. Soc. (London) **344**, 441 (1975).