Physics 6346, Electromagnetic Theory I
Fall 2000
Homework 1 Solutions (Revised 9/5/00)

1. **Review of vector calculus.** Prove the following identities from vector calculus:

   (a) $\nabla \times \nabla \psi = 0$

   (b) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

   (c) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

   (d) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

   Solution. This is standard material. If you had trouble, please review a textbook on mathematical methods, such as *Arfken* or *Boas*.

2. **Curvilinear coordinates.** We will often work in coordinate systems other that rectangular coordinates—for instance, in cylindrical or spherical coordinates. Explicit forms of vector operations in these coordinate systems are on the inside back cover of *Jackson*. Here we'll review how this is done. A good discussion can be found in *Morse and Feshbach*, §1.3.

   We want to go from the rectangular coordinates (x, y, z) to a new set of coordinates (ξ_1, ξ_2, ξ_3). An infinitesimal displacement $d\mathbf{r}$ along the curve $\mathbf{r}(\xi_1, \xi_2, \xi_3)$ is given by

   $$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \xi_1} d\xi_1 + \frac{\partial \mathbf{r}}{\partial \xi_2} d\xi_2 + \frac{\partial \mathbf{r}}{\partial \xi_3} d\xi_3. \tag{1}$$

   If we vary ξ_1 while holding ξ_2 and ξ_3 fixed, then $\partial \mathbf{r}/\partial \xi_1$ is tangent to $\mathbf{r}$ along this curve; likewise for ξ_2 and ξ_3. If $\mathbf{e}_1$ is a unit vector along this direction, then we can write $\partial \mathbf{r}/\partial \xi_1 = h_1 \mathbf{e}_1$, with $h_1 = |\partial \mathbf{r}/\partial \xi_1|$; similarly, $\partial \mathbf{r}/\partial \xi_2 = h_2 \mathbf{e}_2$, with $h_2 = |\partial \mathbf{r}/\partial \xi_2|$, and $\partial \mathbf{r}/\partial \xi_3 = h_1 \mathbf{e}_3$, with $h_3 = |\partial \mathbf{r}/\partial \xi_3|$. The quantities (h_1, h_2, h_3) are called *scale factors*. If $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are orthogonal at each point in space, then we have an *orthogonal* coordinate system. In this coordinate system an element of arclength ds is given by

   $$ds^2 = dx^2 + dy^2 + dz^2 = h_1^2 d\xi_1^2 + h_2^2 d\xi_2^2 + h_3^2 d\xi_3^2, \tag{2}$$

   where the scale factors h_n are given by

   $$h_n^2 = \left(\frac{\partial x}{\partial \xi_n}\right)^2 + \left(\frac{\partial y}{\partial \xi_n}\right)^2 + \left(\frac{\partial z}{\partial \xi_n}\right)^2. \tag{3}$$

   In terms of the scale factors, the volume element in the new coordinate system is

   $$dV = h_1 h_2 h_3 d\xi_1 \, d\xi_2 \, d\xi_3, \tag{4}$$

the gradient of a scalar function Φ is

$$\nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial\xi_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial\xi_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial\xi_3}\mathbf{e}_3, \tag{5}$$

and the Laplacian of Φ is

$$\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\xi_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial\xi_1}\right) + \frac{\partial}{\partial\xi_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial\xi_2}\right) + \frac{\partial}{\partial\xi_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial\xi_3}\right)\right]. \tag{6}$$

Find dV, $\nabla\Phi$, and $\nabla^2\Phi$ in the following coordinate systems:

(a) Cylindrical: $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$.

(b) Spherical: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

(c) Parabolic cylindrical coordinates: $x = (1/2)(u^2 - v^2)$, $y = uv$, $z = z$.

Solution. The results for cylindrical and spherical coordinates can be found in textbooks and *Jackson.* For parabolic cylinder coordinates, we have $(\xi_1, \xi_2, \xi_3) = (u, v, z)$, with the scale factors $h_1 = h_2 = \sqrt{u^2 + v^2}$, $h_3 = 1$, and orthogonal unit vectors $(\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_z)$. Then $dV = (u^2 + v^2)du\,dv\,dz$, and

$$\nabla\Phi = \frac{1}{\sqrt{u^2+v^2}}\left(\frac{\partial\Phi}{\partial u}\mathbf{e}_u + \frac{\partial\Phi}{\partial v}\mathbf{e}_v\right) + \frac{\partial\Phi}{\partial z}\mathbf{e}_z, \tag{7}$$

$$\nabla^2\Phi = \frac{1}{u^2+v^2}\left(\frac{\partial^2\Phi}{\partial u^2} + \frac{\partial^2\Phi}{\partial v^2}\right) + \frac{\partial^2\Phi}{\partial z^2}. \tag{8}$$

3. **Dirac delta function.**

(a) *Jackson's* Problem 1.2 gives a possible representation of the Dirac δ function, in terms of a Gaussian. Give at least two other representations (you may have already encountered the square box, the Lorentzian, and the sinc function, but you can also be creative). Make sure to normalize your function properly.

Solution. For the Gaussian, we have

$$\delta(x) = \frac{1}{\sqrt{2\pi}\epsilon}e^{-x^2/2\epsilon^2}, \tag{9}$$

in the limit that $\epsilon \to 0$. If you plot this you'll see that the function becomes narrower and higher as $\epsilon \to 0$. You can also try the Lorentzian,

$$\delta(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}, \tag{10}$$

or the sinc function,

$$\delta(x) = \frac{1}{\pi}\frac{\sin(x/\epsilon)}{x}, \tag{11}$$

with $\epsilon \to 0$ in both cases.

(b) Using the result of *Jackson's* 1.2, express the three dimensional δ function in cylindrical and spherical coordinates.

Solution. If you work through *Jackson* 1.2, you'll find that in an orthogonal coordinate system (ξ_1, ξ_2, ξ_3), the Dirac delta function can be written as

$$\delta^{(3)}(\mathbf{x} - \mathbf{x}') = \frac{1}{h_1 h_2 h_3} \delta(\xi_1 - \xi_1')\delta(\xi_2 - \xi_2')\delta(\xi_3 - \xi_3'). \tag{12}$$

In cylindrical coordinates, $(\xi_1, \xi_2, \xi_3) = (\rho, \phi, z)$, and $h_1 = 1$, $h_2 = \rho$, and $h_3 = 1$, so that

$$\delta^{(3)}(\mathbf{x} - \mathbf{x}') = \frac{\delta(\rho - \rho')\,\delta(\phi - \phi')\,\delta(z - z')}{\rho}. \tag{13}$$

In spherical coordinates, $(\xi_1, \xi_2, \xi_3) = (r, \theta, \phi)$, and $h_1 = 1$, $h_2 = r$, and $h_3 = r \sin\theta$, so that

$$\delta^{(3)}(\mathbf{x} - \mathbf{x}') = \frac{\delta(r - r')\,\delta(\theta - \theta')\,\delta(\phi - \phi')}{r^2 \sin\theta}. \tag{14}$$

This is often written in the following form:

$$\delta^{(3)}(\mathbf{x} - \mathbf{x}') = \frac{\delta(r - r')\,\delta(\cos\theta - \cos\theta')\,\delta(\phi - \phi')}{r^2}. \tag{15}$$

See p. 120 in *Jackson*.

4. **Simple applications of Gauss's law.** The following charge distributions are highly symmetric; the electric fields produced by them are easily calculated using Gauss's law in integral form. You should be able to do these in your sleep (of course, as a first year graduate student you're probably not sleeping much). Calculate the electric field (magnitude and direction) in each case, being careful to spell out all steps.

(a) A point charge q.

Solution. These are all standard exercises, which can be found in any introductory text, so I'll simply quote the result. If you had difficulty, please review and see me. For a point charge,

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{e}_r}{r^2}. \tag{16}$$

(b) A sphere of radius R with a uniform charge density ρ (find the field both inside and outside the sphere).

Solution.

$$\mathbf{E}(\mathbf{x}) = \begin{cases} (\rho r/3\epsilon_0)\mathbf{e}_r & r < R \\ (\rho R^3/3\epsilon_0 r^2)\mathbf{e}_r & r > R. \end{cases} \tag{17}$$

(c) An infinite line of charge with charge per unit length λ.

Solution.

$$\mathbf{E}(\mathbf{x}) = \frac{\lambda}{2\pi\epsilon_0 r}\mathbf{e}_r. \tag{18}$$

(d) An infinite cylinder of radius R with a uniform charge density ρ (find the field both inside and outside the cylinder).

Solution.

$$\mathbf{E}(\mathbf{x}) = \begin{cases} (\rho r/2\epsilon_0)\mathbf{e}_r & r < R \\ (\rho R^2/2\epsilon_0 r)\mathbf{e}_r & r > R. \end{cases} \tag{19}$$

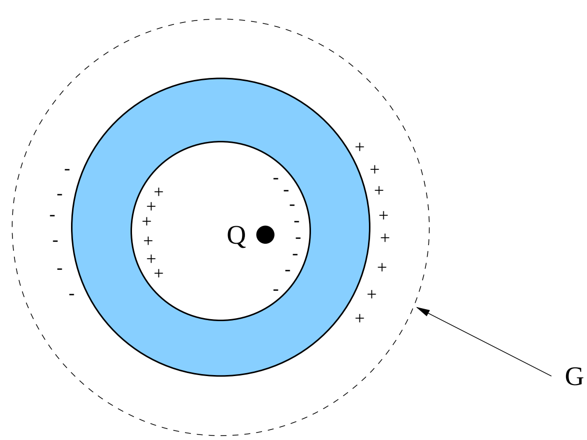
5. Use Gauss's law to prove the following statements about conductors (essentially *Jackson* 1.1):

(a) Any excess charge placed on a conductor must lie entirely on its surface.

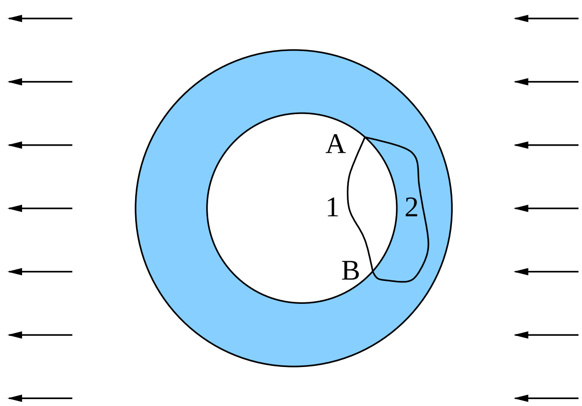
Solution. First, recall that in equilibrium the electric field inside a conductor is zero. Why? Suppose that the field were initially nonzero; any charges in the interior would then move in response to the field (since this is a conductor). After some relaxation time this process stops, since the moving charges produce currents which dissipate energy. The final configuration is one in which the charges have been arranged so that the field in the interior is zero. Since $\mathbf{E} = 0$ everywhere inside the conductor, from Gauss's Law the charge density $\rho = 0$ everywhere in the interior. Therefore any excess charge can only reside on the surface of the conductor.

(b) A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed within it.

Solution. Let's start with the second part. Consider a positive charge Q placed inside a hollow conductor as shown in the figure below. The charge induces a charge density on the interior surface of the conductor in such a way that the electric field in the interior of the conductor is zero (the net charge on the interior surface must be $-Q$). Assuming that the conductor is charge neutral, this means that there is an induced charge density on the exterior surface of total charge Q. If we apply Gauss's Law to the Gaussian surface G surrounding the conductor, the total charge enclosed is still Q, and there is therefore an electric field outside the conductor.



Next, consider some charge exterior to the conductor, which produces an electric field, as shown sketched in the figure. The electric field in the conductor is zero, with induced charge densities on the exterior and interior surfaces of the conductor. Now imagine moving a charge on the interior surface from point A to point B along path 2 which goes through the conductor itself. Since $\mathbf{E} = 0$ in the conductor, $\int_2 \mathbf{E} \cdot d\mathbf{l} = 0$ along this path. Next, move the same charge from A to B along path 1, in the interior cavity of the conductor. Since the electrostatic field is conservative, the line integral $\int_1 \mathbf{E} \cdot d\mathbf{l} = 0$ along this path also. In fact, this must be true for any path which we chose in the interior, so we have quite generally $\mathbf{E} = 0$ in the interior—the conductor shields its interior from fields due to charges placed outside. This is the principle behind the *Faraday cage*.



(c) The electric field at the exterior surface of a charged conductor is normal to the surface and has a magnitude $\sigma(\mathbf{x})/\epsilon_0$, where $\sigma(\mathbf{x})$ is the local surface charge density.

Solution. First, we note that in equilibrium the field at exterior surface must be normal to the surface—a component tangent to the surface would cause charges to move on the surface, until they had arranged themselves so that the tangential component is zero. Once we've established this result, the magnitude of the field is derived by using Gauss's Law with a Gaussian "pillbox" which cuts through the surface. The electric field is zero on the conducting side of the pillbox, so $\oint \mathbf{E} \cdot \mathbf{n} da = EA$, with A the area on the surface (note that for a sheet of charge this becomes $2EA$, where the factor of two comes from the two side of the sheet). Setting this equal to Q/ϵ_0, and then dividing by A, we obtain the desired result.