Chapter 6

Complex Variable Methods

In two dimensions there are elegant methods for solving electrostatics problems which take advantage of the fact that analytic functions in the complex $z = x + iy$ plane are *harmonic*—they are solutions of Laplace's equation. The use of these techniques is almost an art form, and will only be briefly touched upon in the notes. For further details, I can recommend G. F. Carrier, M. Krook, and C. E. Pearson, *Functions of a Complex Variable*.

6.1 Analytic Functions and the Cauchy-Riemann Equations

Let $f(z)$ be a complex function of the complex variable $z = x + iy$. A complex function $f(z)$ is *analytic* at the point z_0 if its derivative df/dz exists at $z = z_0$ and at each point in some neighborhood of z_0. The derivative is defined similarly to the derivative of a function of a real variable,

$$f'(z_0) \equiv \left(\frac{df}{dz}\right)_{z=z_0} = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{6.1}$$

Most common functions are differentiable, and therefore analytic; for instance, z^n, e^z, $\ln z$, and so on. However, since the limit in Eq. (6.1) must be the same *for any mode of approach*, differentiability for functions of a complex variable is much more restrictive than for functions of a real variable. For instance, the function $f(z) = zz^* = x^2 + y^2$ is *not* differentiable. To see this, consider the limit

$$\lim_{z\to z_0} \frac{zz^* - z_0 z_0^*}{z - z_0} = \lim_{z\to z_0} \left(z^* + z_0 \frac{z^* - z_0^*}{z - z_0}\right), \tag{6.2}$$

and let z approach z_0 from a direction making a fixed angle θ with the real axis, so that $z - z_0 = re^{i\theta}$, with $r \to 0$. Then the limit is

$$\lim_{z\to z_0} \frac{zz^* - z_0 z_0^*}{z - z_0} = z_0^* + z_0 e^{-2i\theta}, \tag{6.3}$$

which depends on θ; i.e., the derivative depends on the direction of approach to the point z.

Analytic functions have an amazing property—their real and imaginary parts satisfy the *Cauchy-Riemann equations*. To see how this works, let's call the real and imaginary parts of $f(z)$ u and v; i.e., $f = u + iv$, with u and v real. The derivative of this function with respect to z is

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{x \to x_0, y \to y_0} \frac{[u(x,y) - u(x_0,y_0)] + i[v(x,y) - v(x_0,y_0)]}{(x - x_0) + i(y - y_0)}. \end{aligned} \tag{6.4}$$

If this derivative exists, then it should be independent of the mode of approach to the point z_0. For instance, we could take $y = y_0$ and then take the limit as x approaches x_0:

$$\begin{aligned} f'(z_0) &= \lim_{x \to x_0} \frac{[u(x,y_0) - u(x_0,y_0)] + i[v(x,y_0) - v(x_0,y_0)]}{(x - x_0)} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \end{aligned} \tag{6.5}$$

Alternatively, we could have $x = x_0$ and take the limit in y:

$$\begin{aligned} f'(z_0) &= \lim_{y \to y_0} \frac{[u(x_0,y) - u(x_0,y_0)] + i[v(x_0,y) - v(x_0,y_0)]}{i(y - y_0)} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned} \tag{6.6}$$

By equating the real and imaginary parts of these expressions, we obtain the *Cauchy-Riemann* equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{6.7}$$

An analytic function will satisfy the Cauchy-Riemann equations. They can be combined to obtain Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{6.8}$$

Therefore the real and imaginary parts of an analytic function are "harmonic"; for an analytic function $F(z)$, we have $\nabla^2 F = 0$, so that analytic functions are solutions of Laplace's equation!

Some terminology:

- Analytic functions are often referred to as *holomorphic*.
- An *entire* function is analytic for all finite z; examples would be polynomials in z and e^z.

- A *meromorphic* function is one whose only singularities are poles.
- Functions such as $z^{1/2}$ are *multi-valued*; we can insure a consistent definition of such a function by placing a *branch cut* in the complex plane, which terminates at a *branch point*. For $z^{1/2}$, the branch point is at $z = 0$, and we can place the cut along the negative real axis. Such functions are extremely important in applications, as the cuts are often associated with conducting surfaces.

For applications in electrostatics, we can introduce the complex "potential" $w(z)$, such that $\Phi = \mathcal{R}(w)$, and $\Psi = \mathcal{I}(w)$, with Φ the potential and Ψ the "stream function" (terminology borrowed from fluid mechanics); the lines of constant Φ are then the equipotentials, and the lines of constant Ψ are the field lines (or the streamlines). By combining the Cauchy-Riemann equations once again, we can show that

$$\frac{\partial \Phi}{\partial x}\frac{\partial \Psi}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial \Psi}{\partial y} = \nabla\Phi \cdot \nabla\Psi = 0, \tag{6.9}$$

which shows that the streamlines and the field lines are mutually orthogonal. For example, suppose $w(z) = z^2$; then separating into real and imaginary parts, we find

$$w = x^2 - y^2 + i2xy, \tag{6.10}$$

so that $\Phi = x^2 - y^2$ and $\Psi = 2xy$. Lines of constant Φ (the equipotentials) give hyperbolae; lines of constant Ψ (the field lines) also give hyperbolae, rotated by 45° with respect to the equipotentials.

From the complex potential we can calculate the *complex electric field*, using

$$E = E_x - iE_y = -dw/dz. \tag{6.11}$$

6.2 Some Simple Examples

1. $w(z) = -E_0 z$. Then $\Phi = \mathcal{R}(w) = -E_0 x$, which is the potential due to a uniform field in the x-direction.
2. $w(z) = -(\lambda/2\pi\epsilon_0)\ln z$. Writing $z = \rho e^{i\theta}$, we have $\Phi = -(\lambda/2\pi\epsilon_0)\ln\rho$, which is the potential due to an infinite line charge at the origin.
3. Take two line charges of opposite charge centered at $\pm z_0$; $w(z) = -(\lambda/2\pi\epsilon_0)[\ln(z - z_0) - \ln(z + z_0)]$. Taking the limit that $z_0 \to 0$, we have $w(z) = 2\lambda z_0/2\pi\epsilon_0 z$, which is a two-dimensional point dipole with a dipole moment per unit length $p = 2\lambda z_0$.
4. By taking superpositions of these simple solutions, we can solve more complicated problems. For instance, by adding a point dipole of strength $p = 2\pi\epsilon_0 E_0 a^2$ to the potential due to a uniform field, we have

$$w(z) = -E_0(z - a^2/z), \tag{6.12}$$

which is zero on $|z| = a$, and therefore is the solution to the problem of a grounded conducting cylinder in a uniform electric field. The real part of the potential is

$$\Phi = -E_0(\rho - a^2/\rho)\cos\theta, \tag{6.13}$$

and the stream function (which gives the field lines) is

$$\Psi = -E_0(\rho + a^2/\rho)\sin\theta. \tag{6.14}$$

5. $w(z) = Az^{1/2}$, with A a real constant. It's best to invert this one, so that

$$z = x + iy = \frac{w^2}{A^2} = \frac{\Phi^2 - \Psi^2}{A^2} + i\frac{2\Phi\Psi}{A^2}, \tag{6.15}$$

so that

$$x = \frac{\Phi^2 - \Psi^2}{A^2}, \quad y = \frac{2\Phi\Psi}{A^2}. \tag{6.16}$$

If we want to find the equipotentials, then eliminate Ψ between these two equations:

$$x = \frac{\Phi^2}{A^2} - \frac{A^2}{4\Phi^2}y^2. \tag{6.17}$$

These curves are parabolas tipped on their sides. For $\Phi = 0$, the parabola becomes the negative real axis, so our potential is the the solution to Laplace's equation in the presence of a semi-infinite charged conducting strip along $x < 0$. We can also find the field lines,

$$x = -\frac{\Psi^2}{A^2} + \frac{A^2}{4\Psi^2}y^2. \tag{6.18}$$

These are also parabolas, orthogonal to the equipotentials.

More complicated problems can be solved using conformal mapping methods, which I won't discuss in any detail. However, to illustrate the power of these techniques, I will discuss the solution of one problem, the charged conducting strip.

6.3 The Charged, Grounded, Conducting Strip

Assume that the strip is very thin, so that it can be taken as a line; let's scale the lengths so that the strip is between $x = -1$ and $x = 1$. The problem is to find the potential.

6.3.1 Solution using complex variable methods

The complex potential is

$$w(z) = Ci\sin^{-1}(z), \tag{6.19}$$

with C an as yet undetermined constant. This is an analytic function with suitably defined cuts in the complex z-plane, so it solves Laplace's equation. To see that it satisfies the boundary conditions, it's probably best to invert the function to obtain $z(w)$:

$$z(w) = sin\left(\frac{-iw}{C}\right) = \sin(\Psi/C)\cosh(\Phi/C) - i\cos(\Psi/C)\sinh(\Phi/C), \tag{6.20}$$

where we've used $w = \Phi + i\Psi$ and have written z in terms of its real and imaginary parts. This in turn yields the two equations which determine the field lines and equipotentials,

$$x = \sin(\Psi/C)\cosh(\Phi/C), \quad y = -\cos(\Psi/C)\sinh(\Phi/C). \tag{6.21}$$

Notice that when $\Phi = 0$, $x = \sin(\Psi/C)$ and $y = 0$; as we vary Ψ, x varies between -1 and 1. Therefore, the strip $-1 \leq x \leq 1$ and $y = 0$ is an equipotential, and can be thought of as a charged conducting strip. What about the other equipotentials? From Eq. (6.21), we have

$$\sin(\Psi/C) = \frac{x}{\cosh(\Phi/C)}, \quad \cos(\Psi/C) = \frac{y}{\sinh(\Phi/C)}. \tag{6.22}$$

Adding and squaring, we have

$$\left[\frac{x}{\cosh(\Phi/C)}\right]^2 + \left[\frac{y}{\sinh(\Phi/C)}\right]^2 = 1, \tag{6.23}$$

so we see that the equipotentials are *ellipses*. Similarly, we find for the field lines

$$\left[\frac{x}{\sin(\Psi/C)}\right]^2 - \left[\frac{y}{\cos(\Psi/C)}\right]^2 = 1, \tag{6.24}$$

so the field lines are *hyperbolae*.

From the potential we can calculate the complex electric field,

$$E = E_x - iE_y = -dw/dz = -\frac{iC}{\sqrt{1-z^2}}, \tag{6.25}$$

so that the y component of the field on the strip is

$$E_y = \frac{C}{\sqrt{1-x^2}}\text{sign}(y). \tag{6.26}$$

This is related to the surface charge density, through $\sigma = 2\epsilon_0 E_y$; integrating the surface charge density over the width of the strip, we obtain the charge per unit length λ. Therefore, we find that $C = \lambda/2\pi\epsilon_0$.

Notice that the solution is obtained in a relatively compact form—this is characteristic of solutions obtained using complex variable methods.

6.3.2 Solution using separation of variables in elliptical coordinates

From the form of the equipotentials and field lines obtained above, we might guess that the solution could also have been obtained using separation of variables in elliptical coordinates. We introduce the coordinates (ξ_1, ξ_2) through $x = \cosh\xi_1 \sin\xi_2$, $y = \sinh\xi_1 \cos\xi_2$; then we see that

$$\left(\frac{x}{\cosh\xi_1}\right)^2 + \left(\frac{y}{\sinh\xi_1}\right)^2 = 1, \tag{6.27}$$

$$\left(\frac{x}{\sin\xi_2}\right)^2 - \left(\frac{y}{\cos\xi_2}\right)^2 = 1, \tag{6.28}$$

so that ξ_1 labels ellipses and ξ_2 labels hyperbolae, which are orthogonal. Using these new variables in the Laplacian, after a little algebra we find that Laplace's equation becomes

$$\frac{\partial^2\Phi}{\partial\xi_1^2} + \frac{\partial^2\Phi}{\partial\xi_2^2} = 0. \tag{6.29}$$

The boundary condition on the grounded conducting strip is $\Phi(\xi_1 = 0, \xi_2) = 0$. Using separation of variables, $\Phi(\xi_1, \xi_2) = F(\xi_1)G(\xi_2)$; using the boundary conditions at ∞, we find that $F = \xi_1$, $G =$ constant, so that the solution is $\Phi = C\xi_1$. The final step is to invert the equations which define the elliptical coordinates to find ξ_1 in terms of x and y. The end result is, of course, the same as the result obtained using complex variable methods.

6.4 Edges and Corners

As a final application of complex variable methods, let's consider the behavior of the electrostatic potential near a "corner." The appropriate complex potential is (apart from a multiplicative constant)

$$w(z) = iz^\nu. \tag{6.30}$$

By introducing radial coordinates, $z = \rho e^{i\theta}$, and separating real and imaginary parts, we have

$$\Phi(\rho, \theta) = -\rho^\nu \sin(\nu\theta). \tag{6.31}$$

Now if we want $\Phi = 0$ on $\theta = 0$ and $\theta = \beta$, then we need to choose $\nu\beta = n\pi$, with n a positive integer. Taking $n = 1$, we see that $\nu = \pi/\beta$, and therefore

$$\Phi = -\rho^{\pi/\beta} \sin(\pi\theta/\beta), \tag{6.32}$$

and the electric field near the edge is

$$E_\rho \sim \rho^{\pi/\beta - 1}. \tag{6.33}$$

For $\beta > \pi$, the electric field strength diverges near the edge, and for $\beta = 2\pi$ (a knife edge), it diverges as $\rho^{-1/2}$. Now for a real knife edge this divergence will be cut-off by the thickness of the edge, but the fact remains that sharp edges and points can produce very large electric fields. This is the idea behind the lightning rod—it produces a large field in its vicinity, which may encourage dielectric breakdown of the air, and therefore produce a conducting path through the air to the rod (rather than to your house).