Chapter 7

Legendre Functions and Spherical Harmonics

Laplace's equation can be separated in several coordinate systems, the most important of which are Cartesian (x, y, z), circular cylindrical (ρ, ϕ, z) and spherical polar (r, θ, ϕ). We'll focus on spherical coordinates now, and return to cylindrical coordinates later.

Laplace's equation in spherical coordinates is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0. \tag{7.1}$$

We seek a separable solution of Laplace's equation in the form $\Phi(r, \theta, \phi) = U(r)P(\theta)Q(\phi)/r$; substituting, we find that the equation in the azimuthal angle ϕ is

$$\frac{d^2Q}{d\phi^2} + m^2 Q = 0, \tag{7.2}$$

where m is a separation constant, and the radial equation is

$$\frac{d^2U}{dr^2} - l(l+1)\frac{U}{r^2} = 0, \tag{7.3}$$

with $l(l + 1)$ a second separation constant. The equation in the polar angle θ is more complicated:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0. \tag{7.4}$$

The separation constants m and $l(l + 1)$ are determined with the help of the boundary conditions. The solutions of this and similar eigenvalue problems share many of the following properties:

- A discrete set of eigenvalues
- Series expansions in the appropriate variables

- Recursion relations
- Existence of a generating function

  These properties can be used to compute
- The functions and their derivatives
- Normalization integrals
- Three-function integrals, and others

7.1 Azimuthal and radial eigenfunctions

All the above properties are quite trivial in the case of the eigenfunctions of Eq. (7.2) with the boundary condition of periodicity in $(0, 2\pi)$. The (unnormalized) solutions are

$$U_m(\phi) = \exp(im\phi) \tag{7.5}$$

- The eigenvalues are m^2 with m integer
- The series expansion in ϕ is
  $$U_m(\phi) = \sum_{n=0}^{\infty} \frac{(im\phi)^n}{n!} \tag{7.6}$$
  but it is more convenient to use the variable $X = e^{i\phi}$, in terms of which we simply have
  $$U_m(X) = X^m \tag{7.7}$$
- There is a very simple one-step recursion relation
  $$U_m(X) = XU_{m-1}(X) \tag{7.8}$$
  and the derivative is also given by a simple step-down relation
  $$dU_m/dX = mU_{m-1} \tag{7.9}$$
- The series expansion
  $$\frac{1}{1 - aX} = \sum_{m=0}^{\infty} a^m X^m = \sum_{m=0}^{\infty} a^m U_m(X) \tag{7.10}$$
  shows what is meant by a "generating function"

As for the use of these relations: if we take $U_0 = 1$,we can trivially iterate $U_m(X) = XU_{m-1}(X)$ to get $U_m(X) = X^m$, and the derivative is explicitly given. Normalization is trivial because $\int |U_m|^2 \, d\phi = \int d\phi = 2\pi$. The three-function integrals

$$\int U_l U_m U_n d\phi = 2\pi \delta_{l+m+n,0} \tag{7.11}$$

are also computed directly in this case.

This case is overly simple, because the eigenfunctions obey in effect a first-order differential equation. More typical is the case of

$$\frac{d^2 f}{dx^2} + k^2 f = 0, \tag{7.12}$$

with zero boundary conditions at $x = \pm a/2$ (particle in a box). If we put $\phi = \pi x/a$, this becomes Eq. (7.2) for ϕ in $(-\pi/2, \pi/2)$. Now the eigenfunctions are $U_m = \cos(m\phi)$ with positive half-integer m. For any eigenvalue m^2 there is a "second solution" $\sin(m\phi)$ that does not satisfy the boundary conditions. The recursion relations are now two-step, or involve derivatives. For instance

$$\cos(m+1)\phi = \cos m\phi \cos\phi - \sin m\phi \sin\phi \tag{7.13}$$

$$= \cos m\phi \cos\phi - \frac{\sin^2\phi}{m} \frac{d}{d\cos\phi} \cos m\phi \tag{7.14}$$

The expansion of $\cos m\phi$ in the variable $X = \cos\phi$ is just a polynomial of degree m, the standard Chebyshev polynomial $T_m(X)$ (*Arfken*, page 741). This is just mentioned here as an example to make more palatable the Legendre functions.

Finally, the solutions of the equation

$$\frac{d^2 U}{dr^2} - l(l+1)\frac{U}{r^2} = 0 \tag{7.15}$$

are manifestly of two types: r^{-l}, which for $l > 0$ is well behaved at ∞ but not at the origin, and r^{l+1}, which is well-behaved at the origin but not at ∞ .

7.2 Polar eigenfunctions

7.2.1 Legendre functions

The differential equation in θ is conveniently studied in the variable $x = \cos\theta$, in the interval [-1,1]. For $m = 0$ it is the Legendre equation:

$$\frac{d}{dx}\left[\left(1 - x^2\right)\frac{dP}{dx}\right] + l\,(l+1)\,P = 0. \tag{7.16}$$

To solve, we seek a series solution of the form

$$P(x) = x^\alpha \sum_{j=0}^{\infty} a_j x^j. \tag{7.17}$$

- The usual boundary conditions are that P should remain finite at the end points $x = 1$ and $x = -1$ (corresponding to $\theta = 0$ and $\theta = \pi$). They can be satisfied only if l is a positive integer or zero.

- Without loss of generality we can set $\alpha = 0$. The coefficients a_j of the series solution are then found to obey the recursion relation

$$a_{j+2} = \frac{j(j+1) - l(l+1)}{(j+1)(j+2)} a_j \tag{7.18}$$

- There are two solutions, one even in x, one odd.

  - If l is an even integer or zero, the even solution reduces to the *Legendre polynomial* of order l and is well-behaved at $x = \pm 1$. The "other solution" is not well behaved.
  - If l is an odd integer, the odd solution reduces to the *Legendre polynomial* of order l and is well-behaved at $x = \pm 1$. The "other solution" is not well behaved.

- The first few Legendre polynomials are

$$P_0(x) = 1 \tag{7.19}$$
$$P_1(x) = x \tag{7.20}$$
$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right) \tag{7.21}$$
$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right) \tag{7.22}$$
$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) \tag{7.23}$$

- To prove some properties of the P_l (especially to evaluate integrals) it is useful to know the *Rodrigues' formula*

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2 - 1\right)^l . \tag{7.24}$$

- Even more useful, especially in electrostatics, is the expansion (*generating function*), valid for $|a| < 1$,

$$\frac{1}{\sqrt{1 + a^2 - 2ax}} = \sum_{l=0}^{\infty} a^l P_l(x). \tag{7.25}$$

  If we go back to the variable θ and also put $a = r'/r$, this becomes

$$\frac{r}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta}} = \sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta), \tag{7.26}$$

  or

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta). \tag{7.27}$$

  We recognize an expansion of the Coulomb potential, valid for $r' < r$.

- There are many *recursion relations*. The most useful are probably
  - The step-up and step-down recursions, involving the derivative

$$(l+1)\,P_{l+1} = (l+1)\,xP_l + (x^2-1)P_l' \tag{7.28}$$

$$lP_{l-1} = lxP_l - (x^2-1)P_l' \tag{7.29}$$

  - The two-step recursion

$$(l+1)\,P_{l+1} = (2l+1)\,xP_l - lP_{l-1} \tag{7.30}$$

    which is the most efficient numerical way to generate the P_l

- The orthogonality and normalization integrals are

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\,dx = \frac{2}{2l+1}\,\delta_{ll'} \tag{7.31}$$

- The integrals

$$\int_{-1}^{1} P_l(x)P_m(x)P_n(x)\,dx \tag{7.32}$$

  are also useful, especially in atomic and nuclear physics. It can be shown that they vanish unless

  - $l+m+n$ is even
  - l, m, n satisfy "triangle" inequalities, i.e. $|m-n| \le l \le m+n$

  We probably need them only for $n=1$ and $n=2$.

- By writing $x^2-1 = (x+1)\,(x-1)$ in the Rodrigues formula it is easy to see that

$$P_l(1) = 1, \quad P_l(-1) = (-1)^l. \tag{7.33}$$

  One also can find, using the binomial expansion of $(1-x^2)^l$, that $P_l(0)$ vanishes for l odd (obviously, by parity) and that for even l

$$P_l(0) = (-1)^l \frac{(2l+1)!!}{2^l l!}. \tag{7.34}$$

7.2.2 Linear multipoles

We now run some checks of the above relations, starting from the generating function expansion (7.25), and describe the physical meaning of some of the quantities.

We have already seen that for $r' < r$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta). \tag{7.35}$$

Similarly, by putting $a = r/r'$ in (7.25), we obtain for $r < r'$

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r'}\sum_{l=0}^{\infty}\left(\frac{r}{r'}\right)^l P_l(\cos\theta). \tag{7.36}$$

For $r = r'$ both these expansions converge, except at $\theta = 0$, which of course corresponds to $\mathbf{x} = \mathbf{x}'$. We note that, where the series converges, both sides of these relations satisfy Laplace's equation, because

$$\frac{1}{r^{l+1}}P_l(\cos\theta) \qquad \text{and} \qquad r^l P_l(\cos\theta) \tag{7.37}$$

are the basic separable solutions when there is no dependence on ϕ.

Generally, a ϕ-independent solution of Laplace's equation in the whole region *exterior* to the sphere $r = r'$ has the Legendre expansion

$$\Phi(\mathbf{x}) = \frac{1}{r}\sum_{l=0}^{\infty}C_l\left(\frac{r'}{r}\right)^l P_l(\cos\theta) \tag{7.38}$$

and the coefficients C_l can be found by equating the two sides for $\theta = 0$, where $P_l(1) = 1$, or for some other convenient θ. This is a very useful trick. When $\Phi(\mathbf{x}) = 1/|\mathbf{x}-\mathbf{x}'|$ one quickly finds $C_l = 1$ by this method and one can also turn the argument around to obtain the values of $P_l(-1)$ and $P_l(0)$ already quoted.

The explicit expressions of $P_l(\cos\theta)/r^{l+1}$ for small l are

$$\frac{P_0(\cos\theta)}{r} = \frac{1}{r}, \tag{7.39}$$

$$\frac{P_1(\cos\theta)}{r^2} = \frac{\cos\theta}{r^2} = \frac{z}{r^3} \tag{7.40}$$

$$\frac{P_2(\cos\theta)}{r^3} = \frac{1}{2r^3}\left(3\cos^2\theta - 1\right) = \frac{1}{2r^5}\left(3z^2 - r^2\right). \tag{7.41}$$

We recognize the potential of a dipole oriented along the z axis and of a quadrupole, and we also note that

$$\frac{z}{r^3} = -\frac{\partial}{\partial z}\frac{1}{r} \tag{7.42}$$

$$\frac{1}{r^5}\left(3z^2 - r^2\right) = -\frac{\partial}{\partial z}\frac{z}{r^3} \tag{7.43}$$

We can show that in general

$$-\frac{\partial}{\partial z}\left[\frac{1}{r^{l+1}}P_l(\cos\theta)\right] = \frac{l+1}{r^{l+2}}P_{l+1}(\cos\theta) \tag{7.44}$$

so that the successive application of $-\partial/\partial z$ (at constant x and y) generates the fields of *linear multipoles* consisting of charges aligned along z. As already stated, these are the basic fields for an exterior problem.

Similarly for an *interior* problem the basic fields are $r^l P_l(\cos\theta)$. Their explicit expressions for small l are

$$r^0 P_0(\cos\theta) = 1, \tag{7.45}$$

$$r P_1(\cos\theta) = r\cos\theta = z, \tag{7.46}$$

$$r^2 P_2(\cos\theta) = \frac{1}{2}r^2\left(3\cos^2\theta - 1\right) = \frac{1}{2}\left(3z^2 - r^2\right), \tag{7.47}$$

and we have the "step-down" relations

$$\frac{\partial}{\partial z}\left[r^l P_l(\cos\theta)\right] = l r^{l-1} P_{l-1}(\cos\theta) \tag{7.48}$$

7.2.3 Problems with Azimuthal Symmetry

For problems with azimuthal symmetry (for which $m = 0$), the general solution of Laplace's equation in spherical coordinates is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta). \tag{7.49}$$

As mentioned above, it is often possible to solve problems with azimuthal symmetry by first determining the potential on some convenient axis (the z axis, say), and comparing the result with Eq. (7.49) for $\theta = 0$ (recall that $P_l(1) = 1$). We can then read off the coefficients A_l and B_l. A couple of examples will be included here.

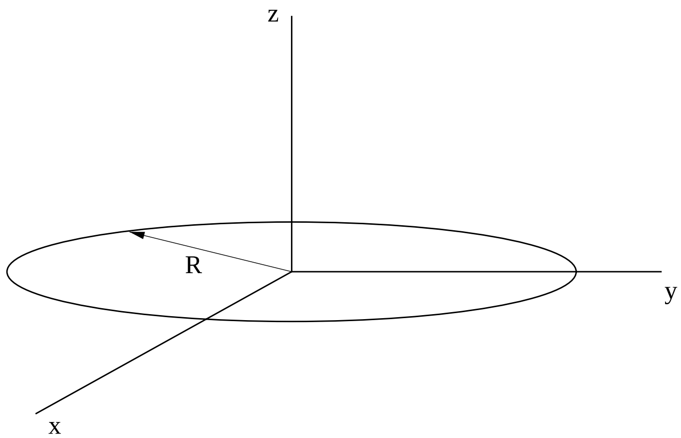
1. *Charged ring of radius R with uniformly distributed charge Q.*

   First, find the potential on the positive z axis. This is

   $$\Phi(z) = \frac{Q}{4\pi\epsilon_0(R^2+z^2)^{1/2}}. \tag{7.50}$$

   Next, expand for $z > R$:

   $$\Phi(z) = \frac{Q}{4\pi\epsilon_0 z}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n n!}(R/z)^{2n}, \tag{7.51}$$



where $(2n-1)!! \equiv (2n-1)\times(2n-3)\times(2n-5)\times\ldots$. Along the positive z axis, Eq. (7.49) becomes

$$\Phi(r=z) = \sum_{l=0}^{\infty}\left[A_l z^l + B_l z^{-(l+1)}\right]. \tag{7.52}$$

Comparing Eqs. (7.51) and (7.52), we see that $A_l = 0$ and

$$B_l = \frac{Q}{4\pi\epsilon_0}(-1)^{l/2}\frac{(l-1)!!}{2^{l/2}(l/2)!}R^l \tag{7.53}$$

for l even. The general solution for $z > R$ is therefore

$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0 r}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n n!}(R/r)^{2n}P_{2n}(\cos\theta). \tag{7.54}$$

For $z < R$ we simply exchange r and R. Therefore, the general solution for all r can be written as

$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n n!}\frac{r_<^{2n}}{r_>^{2n+1}}P_{2n}(\cos\theta), \tag{7.55}$$

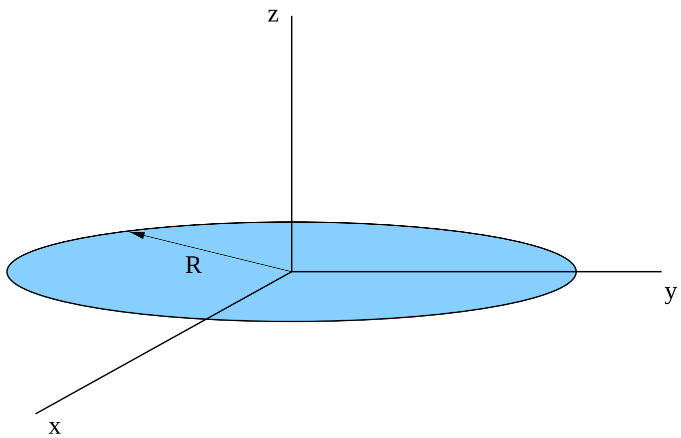
where $r_>$ ($r_<$) is the larger (smaller) of r and R.

2. *Charged conducting disc of radius R* (*Jackson* Problem 3.3).

We are told that the surface charge density is $\sigma(\rho) = k/(R^2-\rho^2)^{1/2}$, with k an undetermined constant (here ρ is the radial distance from the z axis). To find k, integrate over the disc to find the total charge Q:

$$Q = k\int_0^R \frac{2\pi\rho\, d\rho}{(R^2-\rho^2)^{1/2}} = 2\pi k R, \tag{7.56}$$

so $k = Q/2\pi R$.



Next, find the potential along the z-axis by dividing the disc into concentric rings of radius ρ and width $d\rho$; each ring has a charge $dq = \sigma(2\pi\rho\, d\rho)$, so the potential is

$$\begin{aligned}\Phi(z) &= \frac{1}{4\pi\epsilon_0}\int_0^R \frac{(2\pi\rho\, d\rho)\sigma(\rho)}{(\rho^2+z^2)^{1/2}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi R}2\pi\int_0^R \frac{\rho\, d\rho}{(R^2-\rho^2)^{1/2}(\rho^2+z^2)^{1/2}} \\ &= \frac{Q}{8\pi\epsilon_0 R}\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{z^2-R^2}{z^2+R^2}\right)\right]. \qquad (7.57)\end{aligned}$$

Expand this result for $z > R$:

$$\Phi(z) = \frac{Q}{4\pi\epsilon_0 R}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{R}{z}\right)^{2n+1}. \qquad (7.58)$$

The general expansion in spherical coordinates for a problem with azimuthal symmetry is

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l r^l + B_l r^{-(l+1)}\right] P_l(\cos\theta). \qquad (7.59)$$

On the positive z-axis ($\theta = 0$) this becomes

$$\Phi(r,\theta) = \sum_{l=0}^{\infty}\left[A_l z^l + B_l z^{-(l+1)}\right]. \qquad (7.60)$$

Comparing Eq. (7.58) and Eq. (7.60), we can determine the expansion coefficients, and find the final result:

$$\Phi(r,\theta) = \frac{Q}{4\pi\epsilon_0 R}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{R}{r}\right)^{2n+1} P_{2n}(\cos\theta). \qquad (7.61)$$

A similar calculation can be carried out for $z < R$, with the result

$$\Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{\pi}{2}\frac{Q}{R} + \frac{Q}{R}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}\left(\frac{r}{R}\right)^{2n+1} P_{2n+1}(\cos\theta)\right]. \tag{7.62}$$

The potential on the disc is $V = (\pi/2)(Q/4\pi\epsilon_0 R)$, so the capacitance is $C = 8\epsilon_0 R$.

7.3 Associated Legendre Functions and the Spherical Harmonics

For any m, the differential equation in θ is best studied using the variable $x = \cos\theta$, in the interval $[-1, 1]$. It is the associated Legendre equation:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP}{dx}\right] + l\,(l+1)\,P - \frac{m^2}{1-x^2}\,P = 0 \tag{7.63}$$

We consider only the case of integer m, and we can take $m \geq 0$. Then:

- The usual boundary conditions are that P should remain finite at the end points $x = 1$ and $x = -1$ (corresponding to $\theta = 0$ and $\theta = \pi$). They can be satisfied only if l is a positive integer or zero, and $m \leq l$

- The solutions can be written as power series, but it is easier to derive them from the Legendre functions $P_l(x)$ by the formula

$$P_l^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} P_l(x) \tag{7.64}$$

- For any l, m pair there are two solutions, one even in x, one odd. If l is an integer, one solution is a polynomial of order $l - m$ in $x = \cos\theta$ times the factor $\sin^m\theta$. This solution, which has parity $l - m$, is well-behaved at $x = \pm 1$. The "other solution" is not well behaved.

- From the Rodrigues' formula for P_l one obtains at once the *Rodrigues formula* for P_l^m

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}\left(1-x^2\right)^{m/2}\frac{d^{l+m}}{dx^{l+m}}\left(x^2-1\right)^l \tag{7.65}$$

- Clearly $P_l^0(x) = P_l(x)$. Also, from the Rodrigues formula:

$$P_l^l(\cos\theta) = (-1)^l\,\frac{(2l)!}{2^l\,l!}\sin^l\theta \tag{7.66}$$

$$P_l^{l-1}(\cos\theta) = (-1)^{l-1}\,\frac{(2l)!}{2^l\,l!}\sin^l\theta\cos\theta = -\frac{d}{d\theta}P_l^l(\cos\theta) \tag{7.67}$$

- The first few associated Legendre functions with $m > 0$ are

$$P_1^1(x) = -\left(1 - x^2\right)^{1/2} = -\sin\theta \tag{7.68}$$

$$P_2^1(x) = 3x\left(1 - x^2\right)^{1/2} = -3\cos\theta\,\sin\theta \tag{7.69}$$

$$P_2^2(x) = 3\left(1 - x^2\right) = 3\sin^2\theta \tag{7.70}$$

$$P_3^1(x) = -\frac{3}{2}\left(5x^2 - 1\right)\left(1 - x^2\right)^{1/2} = -\frac{3}{2}\left(5\cos^2\theta - 1\right)\sin\theta \tag{7.71}$$

$$P_3^2(x) = 15x\left(1 - x^2\right) = 15\cos\theta\,\sin^2\theta \tag{7.72}$$

$$P_3^3(x) = -15\left(1 - x^2\right)^{3/2} = -15\sin^3\theta \tag{7.73}$$

- Also useful is the expansion (*generating function*), valid for $|a| < 1$,

$$\frac{(2m)!}{2^m m!}\,\frac{(1 - x^2)^{m/2}}{(1 + a^2 - 2ax)^{m+1/2}} = \sum_{l=0}^{\infty} a^l P_{l+m}^m(x) \tag{7.74}$$

  However, it does not have a direct physical meaning.

- There are very many *recursion relations*. Recursions in m are considered more fully in the next section. We list here only

  – the two-step recursion in l, which is the efficient numerical way to generate P_{l+m}^m moving up from P_m^m and P_{m+1}^m

$$(l - m + 1)\,P_{l+1}^m = (2l + 1)\,xP_l^m - (l + m)P_{l-1}^m \tag{7.75}$$

  – the two-step recursion in m, which can generate P_l^m moving down from P_l^l and P_l^{l-1}

$$P_l^{m+1} + \frac{2mx}{\sqrt{1 - x^2}}P_l^m + \left[l(l+1) - m(m+1)\right]P_l^{m-1} = 0 \tag{7.76}$$

- The orthogonality and normalization integrals are

$$\int_{-1}^{1} P_l^m(x)P_{l'}^m(x)\,dx = \frac{2}{2l+1}\,\frac{(l+m)!}{(l-m)!}\,\delta_{ll'} \tag{7.77}$$

  Three-function integrals are considered in the next section.

- Associated Legendre functions for a negative upper index are given by

$$P_l^{-m}(x) = (-1)^m\,\frac{(l+m)!}{(l-m)!}P_l^m(x) \tag{7.78}$$

  With this definition, the Rodrigues formula, recursion relations, and normalization integral are valid for all integer values of m. Note that we have not introduced new eigenfunctions, simply relabeled the ones we had. Why we want to do this is discussed next.

7.3.1 Spherical harmonics

It is often better to deal with the θ and ϕ dependence at the same time by introducing the normalized eigenfunctions

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi} \tag{7.79}$$

Note that Y_l^m and Y_l^{-m} are different eigenfunctions, related by

$$Y_l^{-m} = (-1)^m\, Y_l^{m*} \tag{7.80}$$

These *spherical harmonics* form an orthogonal and complete set. The first few should be memorized, at least in their θ, ϕ dependence.

Multipole fields and "ladder" operations

It is actually easiest to memorize the basic interior solutions, $r^l Y_l^m(\theta,\phi)$, as a function of the Cartesian coordinates x, y, z. For $l = 0$

$$\sqrt{4\pi}\, Y_0^0 = 1 \tag{7.81}$$

For $l = 1$

$$-\sqrt{4\pi}\, rY_1^1 = \sqrt{\frac{3}{2}}\, r\sin\theta\, e^{i\phi} = \sqrt{\frac{3}{2}}\,(x+iy) \tag{7.82}$$

$$\sqrt{4\pi}\, rY_1^0 = \sqrt{3}r\cos\theta = \sqrt{3}\, z \tag{7.83}$$

For $l = 2$

$$\sqrt{4\pi}\, r^2 Y_2^2 = \sqrt{\frac{15}{8}}\, r^2\sin^2\theta\, e^{2i\phi} = \sqrt{\frac{15}{8}}\,(x+iy)^2 \tag{7.84}$$

$$-\sqrt{4\pi}\, r^2 Y_2^1 = \sqrt{\frac{15}{2}}\, r^2\sin\theta\cos\theta\, e^{i\phi} = \sqrt{\frac{15}{2}}\,(x+iy)\, z \tag{7.85}$$

$$\sqrt{4\pi}\, r^2 Y_2^0 = \sqrt{\frac{5}{4}}\, r^2(3\cos^2\theta\, - 1) = \sqrt{\frac{5}{4}}\,\left(3z^2 - r^2\right) \tag{7.86}$$

A look at these expressions suggests that it is convenient to introduce

$$x_+ = x + iy \tag{7.87}$$

$$x_- = x - iy \tag{7.88}$$

with the inverse transformation

$$x = \frac{1}{2}\,(x_+ + x_-) \tag{7.89}$$

$$y = \frac{1}{2i}\,(x_+ - x_-) \tag{7.90}$$

which gives

$$\frac{\partial}{\partial x_+} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right) \tag{7.91}$$

$$\frac{\partial}{\partial x_-} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right) \tag{7.92}$$

Then apart from factors, when applied to $r^l Y_l^m(\theta,\phi)$:

- $\frac{\partial}{\partial z}$ steps down from l, m to $l-1, m$
- $\frac{\partial}{\partial x_+}$ steps down from l, m to $l-1, m-1$
- $\frac{\partial}{\partial x_-}$ steps from l, m to $l-1, m+1$
- $z\frac{\partial}{\partial x_+} - x_-\frac{\partial}{\partial z}$ steps down from l, m to $l, m-1$
- $x_+\frac{\partial}{\partial z} - z\frac{\partial}{\partial x_-}$ steps up from l, m to $l, m+1$

As we did earlier in the case $m = 0$, we can also consider the multipole fields $Y_l^m(\theta,\phi)/r^{l+1}$, which are the basic exterior solutions. The "ladder" operations for these fields are

- $\frac{\partial}{\partial z}$ steps up from l, m to $l+1, m$
- $\frac{\partial}{\partial x_+}$ steps from l, m to $l+1, m-1$
- $\frac{\partial}{\partial x_-}$ steps up from l, m to $l+1, m+1$
- The other two operations, which are very familiar in the quantum theory of angular momentum, have the same effect as before.

The integrals of three spherical harmonics

$$\int Y_l^m Y_{l'}^{m'} Y_{l''}^{m''}\, d\Omega \tag{7.93}$$

are often useful, especially in radiation problems (next semester). It can be shown that they vanish unless

- $m + m' + m'' = 0$ and $l + l' + l''$ is even
- l, l', l'' satisfy "triangle" inequalities, i.e. $|l - l'| \le l'' \le l + l'$

7.3.2 The addition theorem

This is a generalization of the formula

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi') \tag{7.94}$$

for the angle γ between the vectors in the directions of $\mathbf{x}$ and $\mathbf{x}'$. It is

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_l^{m*}(\theta', \phi')Y_l^m(\theta, \phi) \tag{7.95}$$

When inserted in the generating function formula it gives the extremely useful expansion

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_l^{m*}(\theta', \phi')Y_l^m(\theta, \phi) \tag{7.96}$$

where $r_>$ is the larger of r, r'.

7.4 Fields in a hole or near a point

The field is very strong near a sharp conducting point and does not penetrate well inside holes, with a power-law behavior similar to that near edges and inside corners. Near edges and in corners (2-d) we found $\Phi \propto \rho^\alpha$, $E \propto \rho^{\alpha-1}$ with $\alpha = \pi/\theta_0$; near conical points and in conical holes (3-d) one finds $\Phi \propto r^\nu$, $E \propto r^{\nu-1}$, but the relation between ν and the cone angle β (see *Jackson* Fig. 3.5) is not as simple as $\nu = \pi/2\beta$, where 2β in 3-d is the same as θ_0 in 2-d. However, for a hole ($0 < \beta < \pi/2$) there is the excellent approximation (see *Jackson* Fig. 3.6)

$$v \simeq \frac{x_{01}}{\beta} - \frac{1}{2} - c_0\,\beta \tag{7.97}$$

where x_{01} is the first zero of the Bessel function $J_0(x)$ and c_0 is chosen so that $\nu = 1$ for a flat surface ($\beta = \pi/2$), $c_0 = 4x_{01}/\pi^2 - 3/\pi$. Numerically, $x_{01} \simeq 2.405$ and $c_0 \simeq 0.01978$. Actually, this approximation is also good for a point that is not too sharp and fails only when $\pi - \beta < 10\,\text{deg}$, in which case

$$\nu \simeq \left(2\ln\frac{2}{\pi - \beta}\right)^{-1} \tag{7.98}$$

To obtain these results one must construct solutions of the Legendre equation that are regular at the North pole ($\theta = 0$) and are not necessarily regular at the South pole ($\theta = \pi$). These solutions are known as the *Legendre functions* $P_\nu(x)$, where $x = \cos\theta$: they are conventionally normalized by setting $P_\nu(1) = 1$ and reduce to the Legendre polynomials for integer ν. For a given β, one picks ν so that $\cos\beta$ is the first zero of $P_\nu(x)$. The dominant behavior near the apex is then

$$\Phi = A_\nu r^\nu P_\nu(\cos\theta) \tag{7.99}$$

and the field components are given by *Jackson* Eq. (3.46). There is a simple expansion of P_ν in powers of the variable $(1-x)/2 = \sin^2(\theta/2)$ that is quite useful for the fields in shallow holes (ν near 1). The recursion relations for the Legendre polynomials are also valid for the Legendre functions, so in practice we need only tables of $P_\nu(x)$ for $0 < \nu < 1$.