

Chapter 15

Conservation of Energy and Momentum

We go directly to the law of conservation of energy, mechanical and electromagnetic, for a system of particles that interact by electromagnetic forces and are also subject to external electromagnetic fields. Then we examine more in detail the magnetic energy.

15.1 Energy density and flow

We assume (for the time being) that only electromagnetic forces are present and that particle motions are confined to a finite volume V . At speeds small compared to c , the particles obey Newton's equations

$$m_i \frac{d\mathbf{v}_i}{dt} = q_i \mathbf{E}(\mathbf{x}_i) + \mathbf{v}_i \times \mathbf{B}(\mathbf{x}_i) \quad (15.1)$$

Multiplying both sides by \mathbf{v}_i and summing over i we obtain

$$\sum_i \frac{d}{dt} \left(\frac{1}{2} m_i v_i^2 \right) = \sum_i q_i \mathbf{v}_i \cdot \mathbf{E}(\mathbf{x}_i) \quad (15.2)$$

or

$$\frac{d}{dt} U_{kin} = \int_V \mathbf{J}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) d^3x \quad (15.3)$$

where we have introduced the total kinetic energy

$$U_{kin} = \sum_i \left(\frac{1}{2} m_i v_i^2 \right) \quad (15.4)$$

and the current density

$$\mathbf{J}(\mathbf{x}) = \sum_i q_i \mathbf{v}_i \delta(\mathbf{x} - \mathbf{x}_i) \quad (15.5)$$

Equation (15.3) says that $\mathbf{J} \cdot \mathbf{E}$ is the power per unit volume supplied by the field to the particles. We note that \mathbf{E} is the total field, including the field generated by the particles

themselves. It is convenient to introduce the kinetic energy density

$$u_{kin}(\mathbf{x}) = \sum_i \left(\frac{1}{2} m_i v_i^2 \right) \delta(\mathbf{x} - \mathbf{x}_i) \quad (15.6)$$

such that $U_{kin} = \int_V u_{kin}(\mathbf{x}) d^3x$. Then Eq. (15.3) can be written

$$\int_V \left[\frac{\partial}{\partial t} u_{kin}(\mathbf{x}) - \mathbf{J}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) \right] d^3x = 0 \quad (15.7)$$

We now obtain another expression for $\mathbf{J} \cdot \mathbf{E}$ from Maxwell's equations. We multiply by $\mathbf{B} \cdot$ both sides of Faraday's law and obtain

$$\mathbf{B} \cdot \nabla \times \mathbf{E} + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (15.8)$$

Similarly, we multiply by $\mathbf{E} \cdot$ both sides of the Ampère-Maxwell law and obtain

$$\mathbf{E} \cdot \nabla \times \mathbf{B} - \mu_0 \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \cdot \mathbf{E} \quad (15.9)$$

Subtracting side by side and using $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{B}$ we find

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) + \frac{\mu_0}{2} \frac{\partial}{\partial t} (\epsilon_0 E^2 + B^2 / \mu_0) = -\mu_0 \mathbf{J} \cdot \mathbf{E} \quad (15.10)$$

This is the equation we were looking for. We rewrite it as

$$\nabla \cdot \mathbf{S} + \frac{\partial u_{em}}{\partial t} = -\mathbf{J} \cdot \mathbf{E}, \quad (15.11)$$

having defined the important quantities

$$\mathbf{S} = \mathbf{E} \times \mathbf{B} / \mu_0 \quad (15.12)$$

$$u_{em} = \frac{1}{2} (\epsilon_0 E^2 + B^2 / \mu_0) \quad (15.13)$$

It is easy to see that u_{em} is the electromagnetic energy density and that the *Poynting vector* \mathbf{S} represents the energy flow: combine (15.3) and (15.11) to obtain

$$\nabla \cdot \mathbf{S} + \frac{\partial}{\partial t} (u_{em} + u_{kin}) = 0 \quad (15.14)$$

which is of the standard form of an equation of continuity for the *total* energy density $u_{em} + u_{kin}$. Integrating over the volume V bounded by the surface S with outward-pointing normal \mathbf{n} :

$$\oint_S \mathbf{S} \cdot \mathbf{n} da + \frac{\partial U}{\partial t} = 0 \quad (15.15)$$

where $U = U_{em} + U_{kin}$ and U_{em} is the electromagnetic energy contained in volume V

$$U_{em} = \int_V u_{em} d^3x = \int_V \frac{1}{2} (\epsilon_0 E^2 + B^2 / \mu_0) d^3x \quad (15.16)$$

15.2 Conservation of Momentum

The total force on a collection of charged particles is

$$\mathbf{F} = \int d^3x (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}). \quad (15.17)$$

Using Newton's Second Law, we have

$$\frac{d\mathbf{P}_{\text{mech}}}{dt} = \int d^3x (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}). \quad (15.18)$$

Use Maxwell's equations to eliminate the sources in favor of the fields:

$$\frac{d\mathbf{P}_{\text{mech}}}{dt} = \int d^3x \left[\epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \frac{1}{\mu_0} \left(\nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \times \mathbf{B} \right]. \quad (15.19)$$

Next, use

$$\begin{aligned} \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} &= \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) - \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} \\ &= \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times (\nabla \times \mathbf{E}), \end{aligned} \quad (15.20)$$

to write

$$\begin{aligned} \frac{d\mathbf{P}_{\text{mech}}}{dt} &= \int d^3x [\epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} - \epsilon_0 \mathbf{E} \times (\nabla \times \mathbf{E}) + (\nabla \cdot \mathbf{B}) \mathbf{B} / \mu_0 - (\mathbf{B} / \mu_0) \times (\nabla \times \mathbf{B}) \\ &\quad - \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B})]. \end{aligned} \quad (15.21)$$

Moving the time derivative over to the left hand side, we have

$$\begin{aligned} \frac{d\mathbf{P}_{\text{mech}}}{dt} + \frac{d}{dt} \int d^3x (\mathbf{E} \times \mathbf{B} / \mu_0) &= \int d^3x [\epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} - \epsilon_0 \mathbf{E} \times (\nabla \times \mathbf{E}) \\ &\quad + (\nabla \cdot \mathbf{B}) \mathbf{B} / \mu_0 - (\mathbf{B} / \mu_0) \times (\nabla \times \mathbf{B})]. \end{aligned} \quad (15.22)$$

Now work on the right hand side; a typical term is

$$\begin{aligned} [(\nabla \cdot \mathbf{E}) \mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})]_{\alpha} &= (\partial_{\beta} E_{\beta}) E_{\alpha} - \epsilon_{\alpha\beta\gamma} E_{\beta} \epsilon_{\gamma\delta\lambda} \partial_{\delta} E_{\lambda} \\ &= E_{\alpha} \partial_{\beta} E_{\beta} - E_{\beta} \partial_{\alpha} E_{\beta} + E_{\beta} \partial_{\beta} E_{\alpha}, \end{aligned} \quad (15.23)$$

where we've used $\epsilon_{\alpha\beta\gamma} \epsilon_{\gamma\delta\lambda} = \delta_{\alpha\delta} \delta_{\beta\lambda} - \delta_{\alpha\lambda} \delta_{\beta\delta}$. This can finally be written as a total derivative:

$$[(\nabla \cdot \mathbf{E}) \mathbf{E} - \mathbf{E} \times (\nabla \times \mathbf{E})]_{\alpha} = \partial_{\beta} (E_{\alpha} E_{\beta} - \frac{1}{2} E^2 \delta_{\alpha\beta}). \quad (15.24)$$

Then, introducing the field momentum,

$$\mathbf{P}_{\text{field}} = \int_V d^3x \mathbf{E} \times \mathbf{B} / \mu_0, \quad (15.25)$$

and the *Maxwell stress tensor*,

$$T_{\alpha\beta} = \epsilon_0 E_\alpha E_\beta + B_\alpha B_\beta / \mu_0 - \frac{1}{2}(\epsilon_0 E^2 + B^2 / \mu_0) \delta_{\alpha\beta}, \quad (15.26)$$

we have

$$\begin{aligned} \frac{d}{dt}(\mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{field}})_\alpha &= \int_V d^3x \frac{\partial T_{\alpha\beta}}{\partial x_\beta} \\ &= \oint_S T_{\alpha\beta} n_\beta da. \end{aligned} \quad (15.27)$$

We see that $T_{\alpha\beta} n_\beta$ is the α^{th} component of the momentum flux across the surface S into the volume V .