PHZ 3113 Fall 2017

Homework #3, Due Friday, September 15

1. *The variables x and y are related to the variables u and v by $x = e^u \cos v$, $y = e^u \sin v$. Write the Laplacian operator $\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}$ in the variables u and v.*

Start from equation (5.5) or the equation following (5.21) of Chapter 5,

$$df = f_x\, dx + f_y\, dy,$$

where $f_x = \partial f/\partial x$, *etc.* (not only is this shorthand easier to write, it also makes everything easier to read). Now, write $x = x(u, v)$, $y = y(u, v)$, so that

$$dx = x_u\, du + x_v\, dv, \qquad dy = y_u\, du + y_v\, dv.$$

Thus,

$$\begin{aligned} df = f_x\, dx + f_y\, dy &= f_x(x_u\, du + x_v\, dv) + f_y(y_u\, du + y_v\, dv) \\ &= (f_x x_u + f_y y_u)\, du + (f_x x_v + f_y y_v)\, dv. \end{aligned}$$

So, derivatives of f with respect to u and v are

$$f_u = f_x\, x_u + f_y\, y_u, \qquad f_v = f_x\, x_v + f_y\, y_v.$$

This is just the chain rule, as displayed in equation (5.14); perhaps we could have started from here. Second derivatives are found by repeating the same operations,

$$\begin{aligned} f_{uu} &= (f_x)_u\, x_u + f_x\, x_{uu} + (f_y)_u\, y_u + f_y\, y_{uu} \\ &= (f_{xx}\, x_u + f_{yx}\, y_u)\, x_u + f_x\, x_{uu} + (f_{yx}\, x_u + f_{yy}\, y_u)\, y_u + f_y\, y_{uu} \\ &= f_{xx}\, e^{2u} \cos^2 v + 2 f_{xy}\, e^{2u} \sin v \cos v + f_{yy}\, e^{2u} \sin^2 v + f_x\, e^u \cos v + f_y\, e^u \sin v, \end{aligned}$$

$$\begin{aligned} f_{vv} &= (f_x)_v\, x_v + f_x\, x_{vv} + (f_y)_v\, y_v + f_y\, y_{vv} \\ &= (f_{xx}\, x_v + f_{yx}\, y_v)\, x_v + f_x\, x_{vv} + (f_{yx}\, x_v + f_{yy}\, y_v)\, y_v + f_y\, y_{vv} \\ &= f_{xx}\, e^{2u} \sin^2 v - 2 f_{xy}\, e^{2u} \sin v \cos v + f_{yy}\, e^{2u} \cos^2 v - f_x\, e^u \cos v - f_y\, e^u \sin v. \end{aligned}$$

Thus, $f_{uu} + f_{vv} = e^{2u}\,(f_{xx} + f_{yy})$, or

$$\nabla^2 f = f_{xx} + f_{yy} = e^{-2u}\,(f_{uu} + f_{vv}). \qquad \blacksquare$$

The "magic" in this happens because $(e^u)_{uu} = +e^u$ but $(\cos v)_{vv} = -\cos v$.

2. *The free energy $F(T,V)$ of an ideal gas is*

$$F = -NkT \ln\Big[\frac{V}{N}\left(\frac{mkT}{2\pi\hbar}\right)^{3/2}\Big],$$

where N, k, and $\hbar$ are constants.

(a) From $dF = -S\,dT - p\,dV$, compute S and p. Do you recognize the pressure you obtain?

$$S = -\Big(\frac{\partial F}{\partial T}\Big)_V = Nk\left\{\frac{3}{2} + \ln\left[\frac{V}{N}\Big(\frac{mkT}{2\pi\hbar}\Big)^{3/2}\right]\right\} \qquad \blacksquare$$

$$p = -\Big(\frac{\partial F}{\partial V}\Big)_T = \frac{NkT}{V} \qquad \blacksquare$$

$pV = NkT$ is the ideal gas law. If $N = \nu N_0$, where N_0 is Avogadro's number 6×10^{23}, and $N_0 k = R$, then this is $pV = \nu RT$.

(b) The internal energy is given by $U = F + TS$. Compute U. Your answer might appear to be simple, but show from dU that U should be a function of S and V. Write U as a function of S and V. Compute the pressure from U.

The easy calculation is

$$U = F + TS = -NkT \ln\left[\frac{V}{N}\Big(\frac{mkT}{2\pi\hbar}\Big)^{3/2}\right] + T \times Nk\left\{\frac{3}{2} + \ln\left[\frac{V}{N}\Big(\frac{mkT}{2\pi\hbar}\Big)^{3/2}\right]\right\} = \frac{3NkT}{2}.$$

However, from

$$dU = dF + d(TS) = dF + T\,dS + S\,dT = -S\,dT - p\,dV + T\,dS + S\,dT = T\,dS - pdV,$$

the internal energy is formally a function of S and V, not of T, and so T must be written as a function of S and V,

$$T = \frac{2\pi\hbar}{mk}\left(\frac{N}{V}\right)^{2/3} \exp\Big(\frac{2S}{3Nk} - 1\Big), \qquad U = \frac{3\pi\hbar}{m}\frac{N^{5/3}}{V^{2/3}} \exp\Big(\frac{2S}{3Nk} - 1\Big).$$

Then,

$$p = -\Big(\frac{\partial U}{\partial V}\Big) = \frac{2\,U}{3\,V} = \frac{NkT}{V}.$$

This is still the ideal gas law.

3. *The two constraints $f(x,y,z)=0$ and $g(x,y,z)=0$ in a three-dimensional world define a one-dimensional curve.*

(a) *Find dy/dx along this curve.*

Since $f=0$ and $g=0$, their variations must vanish also,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = f_x\,dx + f_y\,dy + f_z\,dz = 0,$$

$$dg = \frac{\partial g}{\partial x}\,dx + \frac{\partial g}{\partial y}\,dy + \frac{\partial g}{\partial z}\,dz = g_x\,dx + g_y\,dy + g_z\,dz = 0,$$

where f_x is short for $\partial f/\partial x$, etc. Multiply the first by g_z and the second by f_z, and subtract (so that the dz terms cancel):

$$g_z\,(f_x\,dx + f_y\,dy + f_z\,dz) - f_z\,(g_x\,dx + g_y\,dy + g_z\,dz) = (g_z f_x - f_z g_x)\,dx + (g_z f_y - f_z g_y)\,dy = 0.$$

Solve for dy/dx:

$$\frac{dy}{dx} = -\frac{(g_z f_x - f_z g_x)}{(g_z f_y - f_z g_y)}. \qquad \blacksquare$$

That's it.

(b) *Do you get a sensible result for the intersection of the sphere $x^2+y^2+z^2=1$ with the plane $z=0$?*

For $f = x^2+y^2+z^2-1$ and $g=z$, the derivatives are $f_x = 2x$, $f_y = 2y$, $f_z = 2z$, $g_x = 0$, $g_y = 0$, $g_z = 1$; and so

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.$$

Does this make sense? The constraint $x^2+y^2+z^2=1$ is a sphere, and the restriction $z=0$ means x and y live on the circle $x^2+y^2=1$, or $y=\pm\sqrt{1-x^2}$, or

$$\frac{dy}{dx} = \frac{d}{dx}\left(\pm\sqrt{1-x^2}\right) = \mp\frac{x}{\sqrt{1-x^2}} = -\frac{x}{y}. \qquad \blacksquare$$

Just so.

4. *Find the point on the curve defined by* $\frac{5}{8}x^2 - \frac{3}{4}xy + \frac{5}{8}y^2 = 1$ *that is closest to the point* $(x, y) = (1, -1)$.

The (squared) distance from $(1, -1)$ to a point on the plane is $d^2 = (x-1)^2 + (y+1)^2 = f(x, y)$. Minimization of f subject to the constraint $\phi = 0$ is found by the method of Lagrange multipliers (9.7) from

$$F = f + \lambda\phi = (x-1)^2 + (y+1)^2 + \lambda\,(\tfrac{5}{8}x^2 - \tfrac{3}{4}xy + \tfrac{5}{8}y^2 - 1).$$

Extrema or stationary points of F occur where the derivatives vanish,

$$\frac{\partial F}{\partial x} = 2(x-1) + \lambda\,(\tfrac{5}{4}x - \tfrac{3}{4}y) = 0,$$

$$\frac{\partial F}{\partial y} = 2(y+1) + \lambda\,(-\tfrac{3}{4}x + \tfrac{5}{4}y) = 0,$$

$$\frac{\partial F}{\partial \lambda} = \tfrac{5}{8}x^2 - \tfrac{3}{4}xy + \tfrac{5}{8}y^2 - 1 = 0.$$

These must all be satisfied simultaneously. One approach is to start by solving the first two equation for x and y in terms of λ,

$$x = \frac{1}{1+\lambda}, \qquad y = -\frac{1}{1+\lambda}.$$

Then, the constraint $\partial F/\partial\lambda = \phi = 0$ gives two solutions for λ,

$$\frac{2}{(1+\lambda)^2} - 1 = 0, \qquad \lambda_{a,b} = -1 \pm \sqrt{2},$$

which leads back to

$$(x, y)_a = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \qquad (x, y)_b = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}).$$

These are not the only solutions; when $\lambda = -4$ the first two equations are not independent, but both give $3x - 3y + 2 = 0$, or $y = \frac{2}{3} + x$. Then, the constraint $\phi = 0$ gives

$$\frac{1}{2}x^2 + \frac{1}{3}x - \frac{13}{18} = 0, \qquad (x, y)_{c,d} = (\frac{-1 \pm \sqrt{14}}{3}, \frac{1 \pm \sqrt{14}}{3}).$$

These are "extrema," which can be minima, maxima, or saddle points. The distances are

$$d_a^2 = 3 - 2\sqrt{2} = 0.17137, \qquad d_b^2 = 3 + 2\sqrt{2} = 5.82843, \qquad d_c^2 = d_d^2 = \frac{20}{3} = 6.66667.$$

The first of these is the smallest, by far. Thus, the closest point is

$$(x, y)_a = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}).$$

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The figure shows the locus of the constraint, an ellipse with axes at $\pm 45°$, and the four stationary points. Point a is the solution, the minimum of distance, with $d^2 = 3 - 2\sqrt{2} = 0.171$; points c, d are maxima, $d^2 = \frac{20}{3} = 6.667$; and point b, with $d^2 = 3 + 2\sqrt{2} = 5.828$, is a local but not global minimum.