1. The variables $x$ and $y$ are related to the variables $u$ and $v$ by $x = e^u \cos v$, $y = e^u \sin v$.

Write the Laplacian operator $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ in the variables $u$ and $v$.

Start from equation (5.5) or the equation following (5.21) of Chapter 5,

$$df = f_x \ dx + f_y \ dy,$$

where $f_x = \partial f / \partial x$, etc. (not only is this shorthand easier to write, it also makes everything easier to read). Now, write $x = x(u, v)$, $y = y(u, v)$, so that

$$dx = x_u \ du + x_v \ dv, \quad dy = y_u \ du + y_v \ dv.$$

Thus,

$$df = f_x \ dx + f_y \ dy = f_x(x_u \ du + x_v \ dv) + f_y(y_u \ du + y_v \ dv)$$

$$= (f_x x_u + f_y y_u) \ du + (f_x x_v + f_y y_v) \ dv.$$

So, derivatives of $f$ with respect to $u$ and $v$ are

$$f_u = f_x x_u + f_y y_u, \quad f_v = f_x x_v + f_y y_v.$$

This is just the chain rule, as displayed in equation (5.14); perhaps we could have started from here. Second derivatives are found by repeating the same operations,

$$f_{uu} = (f_x)_u \ x_u + f_x \ x_{uu} + (f_y)_u \ y_u + f_y \ y_{uu}$$

$$= (f_{xx} \ x_u + f_{yx} \ y_u) \ x_u + f_x \ x_{uu} + (f_{yx} \ x_u + f_{yy} \ y_u) \ y_u + f_y \ y_{uu}$$

$$= f_{xx} e^{2u} \cos^2 v + 2 f_{xy} e^{2u} \sin v \cos v + f_{yy} e^{2u} \sin^2 v + f_x e^u \cos v + f_y e^u \sin v,$$

$$f_{vv} = (f_x)_v \ x_v + f_x \ x_{vv} + (f_y)_v \ y_v + f_y \ y_{vv}$$

$$= (f_{xx} \ x_v + f_{yx} \ y_v) \ x_v + f_x \ x_{vv} + (f_{yx} \ x_v + f_{yy} \ y_v) \ y_v + f_y \ y_{vv}$$

$$= f_{xx} e^{2u} \sin^2 v - 2 f_{xy} e^{2u} \sin v \cos v + f_{yy} e^{2u} \cos^2 v - f_x e^u \cos v - f_y e^u \sin v.$$

Thus, $f_{uu} + f_{vv} = e^{2u} (f_{xx} + f_{yy})$, or

$$\nabla^2 f = f_{xx} + f_{yy} = e^{-2u} (f_{uu} + f_{vv}).$$

The “magic” in this happens because $(e^u)_{uu} = +e^u$ but $(\cos v)_{vv} = -\cos v$. 

2. The free energy $F(T,V)$ of an ideal gas is

$$F = -NkT \ln \left[ \frac{V}{N} \left( \frac{m k T}{2 \pi \hbar} \right)^{3/2} \right],$$

where $N$, $k$, and $\hbar$ are constants.

(a) From $dF = -S dT - p dV$, compute $S$ and $p$. Do you recognize the pressure you obtain?

$$S = -\left( \frac{\partial F}{\partial T} \right)_V = Nk \left\{ \frac{3}{2} + \ln \left[ \frac{V}{N} \left( \frac{m k T}{2 \pi \hbar} \right)^{3/2} \right] \right\}$$

$$p = -\left( \frac{\partial F}{\partial V} \right)_T = \frac{NkT}{V}$$

$pV = NkT$ is the ideal gas law. If $N = \nu N_0$, where $N_0$ is Avogadro’s number $6 \times 10^{23}$, and $N_0 k = R$, then this is $pV = \nu RT$.

(b) The internal energy is given by $U = F + TS$. Compute $U$. Your answer might appear to be simple, but show from $dU$ that $U$ should be a function of $S$ and $V$. Write $U$ as a function of $S$ and $V$. Compute the pressure from $U$.

The easy calculation is

$$U = F + TS = -NkT \ln \left[ \frac{V}{N} \left( \frac{m k T}{2 \pi \hbar} \right)^{3/2} \right] + T \left\{ \frac{3}{2} + \ln \left[ \frac{V}{N} \left( \frac{m k T}{2 \pi \hbar} \right)^{3/2} \right] \right\} = \frac{3NkT}{2}.$$

However, from

$$dU = dF + d(TS) = dF + T dS + S dT = -S dT - p dV + T dS + S dT = T dS - pdV,$$

the internal energy is formally a function of $S$ and $V$, not of $T$, and so $T$ must be written as a function of $S$ and $V$.

$$T = \frac{2 \pi \hbar}{mk} \left( \frac{N}{V} \right)^{2/3} \exp \left( \frac{2S}{3Nk} - 1 \right), \quad U = \frac{3 \pi \hbar N^{5/3}}{m V^{2/3}} \exp \left( \frac{2S}{3Nk} - 1 \right).$$

Then,

$$p = -\left( \frac{\partial U}{\partial V} \right) = \frac{2U}{3V} = \frac{NkT}{V}.$$

This is still the ideal gas law.
3. The two constraints \( f(x, y, z) = 0 \) and \( g(x, y, z) = 0 \) in a three-dimensional world define a one-dimensional curve.

(a) Find \( \frac{dy}{dx} \) along this curve.

Since \( f = 0 \) and \( g = 0 \), their variations must vanish also,

\[
\begin{align*}
\frac{df}{dx} dx + \frac{df}{dy} dy + \frac{df}{dz} dz &= f_x dx + f_y dy + f_z dz = 0, \\
\frac{dg}{dx} dx + \frac{dg}{dy} dy + \frac{dg}{dz} dz &= g_x dx + g_y dy + g_z dz = 0,
\end{align*}
\]

where \( f_x \) is short for \( \frac{\partial f}{\partial x} \), etc. Multiply the first by \( g_z \) and the second by \( f_z \), and subtract (so that the \( dz \) terms cancel):

\[
g_z (f_x dx + f_y dy + f_z dz) - f_z (g_x dx + g_y dy + g_z dz) = (g_z f_x - f_z g_x) dx + (g_z f_y - f_z g_y) dy = 0.
\]

Solve for \( \frac{dy}{dx} \):

\[
\frac{dy}{dx} = \frac{\left( g_z f_x - f_z g_x \right)}{\left( g_z f_y - f_z g_y \right)}.
\]

That’s it.

(b) Do you get a sensible result for the intersection of the sphere \( x^2 + y^2 + z^2 = 1 \) with the plane \( z = 0 \)?

For \( f = x^2 + y^2 + z^2 - 1 \) and \( g = z \), the derivatives are \( f_x = 2x, f_y = 2y, f_z = 2z, g_x = 0, g_y = 0, g_z = 1; \) and so

\[
\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}.
\]

Does this make sense? The constraint \( x^2 + y^2 + z^2 = 1 \) is a sphere, and the restriction \( z = 0 \) means \( x \) and \( y \) live on the circle \( x^2 + y^2 = 1 \), or \( y = \pm\sqrt{1 - x^2} \), or

\[
\frac{dy}{dx} = \frac{d}{dx} \left( \pm\sqrt{1 - x^2} \right) = \pm\frac{x}{\sqrt{1-x^2}} = -\frac{x}{y}.
\]

Just so.
4. Find the point on the curve defined by \( \frac{5}{8} x^2 - \frac{3}{4} xy + \frac{5}{8} y^2 = 1 \) that is closest to the point \((x, y) = (1, -1)\).

The (squared) distance from \((1, -1)\) to a point on the plane is \( d^2 = (x - 1)^2 + (y + 1)^2 = f(x, y) \). Minimization of \( f \) subject to the constraint \( \phi = 0 \) is found by the method of Lagrange multipliers (9.7) from

\[
F = f + \lambda \phi = (x - 1)^2 + (y + 1)^2 + \lambda \left( \frac{5}{8} x^2 - \frac{3}{4} xy + \frac{5}{8} y^2 - 1 \right).
\]

Extrema or stationary points of \( F \) occur where the derivatives vanish,

\[
\frac{\partial F}{\partial x} = 2(x - 1) + \lambda \left( \frac{5}{4} x - \frac{3}{4} y \right) = 0,
\]

\[
\frac{\partial F}{\partial y} = 2(y + 1) + \lambda \left( -\frac{3}{4} x + \frac{5}{4} y \right) = 0,
\]

\[
\frac{\partial F}{\partial \lambda} = \frac{5}{8} x^2 - \frac{3}{4} xy + \frac{5}{8} y^2 - 1 = 0.
\]

These must all be satisfied simultaneously. One approach is to start by solving the first two equation for \( x \) and \( y \) in terms of \( \lambda \),

\[
x = \frac{1}{1 + \lambda}, \quad y = -\frac{1}{1 + \lambda}.
\]

Then, the constraint \( \frac{\partial F}{\partial \lambda} = \phi = 0 \) gives two solutions for \( \lambda \),

\[
\frac{2}{(1 + \lambda)^2} - 1 = 0, \quad \lambda_{a,b} = -1 \pm \sqrt{2},
\]

which leads back to

\[
(x, y)_a = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \quad (x, y)_b = \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right).
\]

These are not the only solutions; when \( \lambda = -4 \) the first two equations are not independent, but both give \( 3x - 3y + 2 = 0 \), or \( y = \frac{2}{3} + x \). Then, the constraint \( \phi = 0 \) gives

\[
\frac{1}{2} x^2 + \frac{1}{3} x - \frac{13}{18} = 0, \quad (x, y)_{c,d} = \left( \frac{-1 \pm \sqrt{14}}{3}, \frac{1 \pm \sqrt{14}}{3} \right).
\]

These are “extrema,” which can be minima, maxima, or saddle points. The distances are

\[
d_a^2 = 3 - 2\sqrt{2} = 0.17137, \quad d_b^2 = 3 + 2\sqrt{2} = 5.82843, \quad d_c^2 = d_d^2 = \frac{20}{3} = 6.66667.
\]

The first of these is the smallest, by far. Thus, the closest point is

\[
(x, y)_a = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).
\]
The figure shows the locus of the constraint, an ellipse with axes at $\pm 45^\circ$, and the four stationary points. Point $a$ is the solution, the minimum of distance, with $d^2 = 3 - 2\sqrt{2} = 0.171$; points $c$, $d$ are maxima, $d^2 = \frac{20}{3} = 6.667$; and point $b$, with $d^2 = 3 + 2\sqrt{2} = 5.828$, is a local but not global minimum.