1. Define elliptical coordinates $u$, $v$ by $x = \sqrt{u^2 + c^2} \cos v$, $y = u \sin v$.

(a) Write the area element $dA = dx \, dy$ in terms of elliptical $u$, $v$.

The element of area is $dx \, dy = |J| \, du \, dv$, where the Jacobian determinant is

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} \frac{u \cos v}{\sqrt{u^2 + c^2}} & -\frac{\sqrt{u^2 + c^2} \sin v}{\sqrt{u^2 + c^2}} \\ \frac{\sin v}{\sqrt{u^2 + c^2}} & \frac{u \cos v}{\sqrt{u^2 + c^2}} \end{pmatrix} = \frac{u^2}{\sqrt{u^2 + c^2}} \cos^2 v + \sqrt{u^2 + c^2} \sin^2 v = \frac{u^2 + c^2 \sin^2 v}{\sqrt{u^2 + c^2}}.$$ 

(b) Using elliptical coordinates, compute the area of an ellipse with semimajor axis $a$, semiminor axis $b$, where $a^2 = b^2 + c^2$.

It is easy to see that curves of constant $u$ are ellipses,

$$\frac{x^2}{u^2 + c^2} + \frac{y^2}{u^2} = \cos^2 v + \sin^2 v = 1,$$

and $0 < u < b$ and $0 < v < 2\pi$ traces the full ellipse. The area is

$$A = \int_0^b du \int_0^{2\pi} dv \frac{u^2 + c^2 \sin^2 v}{\sqrt{u^2 + c^2}} = \pi \int_0^b du \frac{2u^2 + c^2}{\sqrt{u^2 + c^2}}.$$ 

Since

$$\frac{d}{du} \left( u \sqrt{u^2 + c^2} \right) = \frac{2u^2 + c^2}{\sqrt{u^2 + c^2}},$$

this is

$$A = \left[ \pi u \sqrt{u^2 + c^2} \right]_0^b = \pi b \sqrt{b^2 + c^2} = \pi ab.$$
2. Let the scalar field \( f \) be given by \( f(x, y) = \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} \).

(a) Compute the components of the gradient \( \nabla f \). Where is the derivative troublesome?

The derivatives are
\[
\frac{\partial f}{\partial x} = \frac{x^3 + 3xy^2}{(x^2 + y^2)^{3/2}}, \quad \frac{\partial f}{\partial y} = \frac{3x^2y + y^3}{(x^2 + y^2)^{3/2}},
\]
and
\[
\nabla f = \frac{(x^3 + 3xy^2) \hat{x} - (3x^2y + y^3) \hat{y}}{(x^2 + y^2)^{3/2}}.
\]
This is well-behaved except as \( x^2 + y^2 \to 0 \).

(b) Compute the directional derivative of \( \nabla u f \) in the direction \( u = \hat{x} \cos \alpha + \hat{y} \sin \alpha \). Write your answer in terms of polar coordinates \( \rho, \phi \), where \( x = \rho \cos \phi, y = \rho \sin \phi \). What is the directional derivative along a radial ray (fixed \( \phi \)) as \( \rho \to 0 \)?

The directional derivative is
\[
\frac{df}{du} = u \cdot \nabla f = \frac{\cos \alpha (x^3 + 3xy^2) - \sin \alpha (3x^2y + y^3)}{(x^2 + y^2)^{3/2}}.
\]
In terms of \( \rho \) and \( \phi \), this is
\[
\frac{df}{du} = \cos \alpha (\cos^3 \phi + 3 \cos \phi \sin^2 \phi) - \sin \alpha (3 \cos^2 \phi \sin \phi + \sin^3 \phi)
\]
\[
= \cos \alpha (4 \cos^3 \phi + 3 \cos \phi) - \sin \alpha (3 \sin \phi - 2 \sin^3 \phi).
\]
Either of these is correct, but using \( \cos 3\phi = 4 \cos^3 \phi - 3 \cos \phi, \sin 3\phi = 3 \sin \phi - 4 \sin^3 \phi, \) this can also be written
\[
\frac{df}{du} = \frac{1}{2} [\cos \alpha (3 \cos \phi - \cos 3\phi) - \sin \alpha (3 \sin \phi + \sin 3\phi)] = \frac{3}{2} \cos(\phi + \alpha) - \frac{1}{2} \cos(3\phi - \alpha).
\]
This is perhaps easier to obtain by writing
\[
f = \rho \cos 2\phi, \quad \nabla f = \frac{\partial f}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\phi}, \quad \frac{df}{du} = \cos 2\phi \hat{\rho} \cdot u - 2 \sin 2\phi \hat{\phi} \cdot u.
\]
with
\[
\hat{\rho} \cdot u = \cos \phi \cos \alpha + \sin \phi \sin \alpha = \cos(\alpha - \phi) \quad \hat{\phi} \cdot u = -\sin \phi \cos \alpha + \cos \phi \sin \alpha = \sin(\alpha - \phi),
\]
leading to linear combinations of \( \cos[2\phi \pm (\alpha - \phi)] \), as above.
3. Let the vector field \( \mathbf{v} \) be
\[
\mathbf{v} = 2xy \hat{x} + (x^2 + y^2) \hat{y}.
\]

(a) Compute \( \nabla \cdot \mathbf{v} \). Compute \( \nabla \times \mathbf{v} \).

The divergence is
\[
\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = (2y) + (2y) = 4y
\]

The curl is
\[
\nabla \times \mathbf{v} = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \right) \times (\hat{x} v_x + \hat{y} v_y) = \hat{z} \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) = [(2x) - (2x)] \hat{z} = 0
\]

(b) Compute \( \nabla (\nabla \cdot \mathbf{v}) \) and \( \nabla^2 \mathbf{v} \). Compute \( \nabla \times (\nabla \times \mathbf{v}) \) two different ways.

The gradient of the divergence is
\[
\nabla (\nabla \cdot \mathbf{v}) = \left( \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} \right)(4y) = 4 \hat{y}
\]

The Laplacian is
\[
\nabla^2 \mathbf{v} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left[ 2xy \hat{x} + (x^2 + y^2) \hat{y} \right] = (2 + 2) \hat{y} = 4 \hat{y}
\]

Since \( \nabla \times \mathbf{v} = 0 \), we have \( \nabla \times (\nabla \times \mathbf{v}) = 0 \), also the result of the other way,
\[
\nabla \times (\nabla \times \mathbf{v}) = \nabla (\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} = (4 \hat{y}) - (4 \hat{y}) = 0
\]
4. The electric and gravitational forces between point particles are both inverse square, and in analogy with electrostatics, the gravitational field \( \mathbf{g} \) can be obtained from a gravitational potential \( \Phi = -\nabla \Phi \) that satisfies \( \nabla^2 \Phi = 4\pi G \rho \). In numerical simulations of large-scale cosmological structure, it is sometimes useful to “soften” the Newtonian gravitational potential \( \Phi = -GM/r \) of a point particle of mass \( M \) as

\[
\Phi_a = -\frac{GM}{\sqrt{r^2 + a^2}}.
\]

(a) What is the gravitational field \( \mathbf{g} \) derived from \( \Phi_a \)? How does \( \mathbf{g} \) behave as \( r \to \infty \)? Show that the softening introduces a maximum in the gravitational force between particles.

\[
\mathbf{g} = -\nabla \Phi_a = -\frac{GMr}{(r^2 + a^2)^{3/2}} \dot{\mathbf{r}}.
\]

This behaves as \( GM/r^2 \) as \( r \to \infty \), vanishes linearly as \( r \to 0 \), and has peak magnitude \((2/3\sqrt{3}) GM/a^2 \) at \( r = a/\sqrt{2} \).

(b) The softened \( \Phi_a \) can be interpreted as the potential of a slightly fuzzy particle with a mass density \( \rho_a(r) \). What is \( \rho_a(r) \)? What is the total mass, \( M_a = \int d^3v \rho_a \)? What radius \( r_{1/2} \) contains half the mass?

From Poisson’s equation,

\[
\nabla^2 \Phi_a = \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r} \frac{\partial \Phi}{\partial r} = \frac{3a^2GM}{(r^2 + a^2)^{5/2}} = 4\pi G \rho, \quad \rho_a(r) = \frac{3a^2M}{4\pi (r^2 + a^2)^{5/2}}.
\]

The mass within radius \( r \) is

\[
M_a(r) = \int_0^r 4\pi r'^2 \rho_a(r') = \frac{Mr^3}{(r^2 + a^2)^{3/2}},
\]

and so \( M(r \to \infty) = M \), independent of \( a \). The half-mass radius is

\[
\frac{r_{1/2}^2}{r_{1/2}^2 + a^2} = \frac{1}{2^{2/3}}, \quad r_{1/2} = \frac{a}{\sqrt{2^{2/3} - 1}} = 1.30477 a.
\]

As \( a \to 0 \), this is a mass distribution that has zero density for \( r \neq 0 \), has infinite density as \( r \to 0 \), has extent of order \( a \), and has constant integral total mass. Thus, as \( a \to 0 \) this returns to the point mass, or

\[
\rho_a \to M \delta^{(3)}(\mathbf{r}).
\]