1. Let \( f(x) = \begin{cases} \sin x & 0 < x < \pi, \\ 0 & -\pi < x < 0. \end{cases} \)

(a) Compute the coefficients \( a_n, b_n \) to express \( f(x) \) as a Fourier series. (Be careful with \( n = 1 \).)

For \( n \neq 1 \), the cosine coefficients are

\[
a_n = \frac{1}{\pi} \int_0^\pi \sin x \cos nx \, dx = \frac{1}{\pi} \frac{\cos n\pi}{n^2 - 1} = \begin{cases} \frac{-2/\pi}{n^2 - 1} & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{cases}
\]

(this also holds for \( a_0 \)), and the sine coefficients are

\[
b_n = \frac{1}{\pi} \int_0^\pi \sin x \sin nx \, dx = \frac{\sin n\pi}{n^2 - 1} = 0.
\]

For \( n = 1 \),

\[
a_1 = \frac{1}{\pi} \int_0^\pi \sin x \cos x \, dx = 0, \quad b_1 = \frac{1}{\pi} \int_0^\pi \sin^2 x \, dx = \frac{1}{2}.
\]

Thus,

\[
f(x) = \frac{1}{\pi} + \frac{1}{2} \sin x - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\cos 2kx}{(2k)^2 - 1}.
\]

(b) Compute the value of the sum \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)^2 - 1} \).

Evaluate \( f(x) \) at \( x = \frac{\pi}{2} \):

\[
f(\frac{\pi}{2}) = 1 = \frac{1}{\pi} + \frac{1}{2} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\cos k\pi}{(2k)^2 - 1}.
\]

So,

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k)^2 - 1} = \frac{\pi}{2} \left(1 - \frac{1}{2} - \frac{1}{\pi}\right) = \frac{\pi}{4} - \frac{1}{2}.
\]
(c) Compute the value of the sum $\sum_{k=1}^{\infty} \frac{1}{((2k)^2 - 1)^2}$.

The square of coefficients appears in Parseval's theorem,

$$\int_{0}^{2\pi} f^2(x) = \int_{0}^{\pi} \sin^2 x \, dx = \frac{\pi}{2} = \frac{\pi}{2} a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

and so

$$\frac{1}{2} \left(\frac{2}{\pi}\right)^2 + \left(\frac{1}{2}\right)^2 + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{((2k)^2 - 1)^2} = \frac{1}{2}.$$ 

Thus,

$$\sum_{k=1}^{\infty} \frac{1}{((2k)^2 - 1)^2} = \frac{\pi^2}{4} \left(\frac{1}{2} - \frac{1}{4} - \frac{2}{\pi^2}\right) = \frac{\pi^2}{16} - \frac{1}{2}.$$
2. Because sine and cosine can be written as complex exponentials and vice versa, a Fourier series on the interval \((-\pi, \pi)\) can just as well be written as

\[ f(x) = \sum_{m=-\infty}^{\infty} c_m e^{imx}. \]

(a) Examine the integral \(\int_0^{2\pi} dx e^{-ipx} f(x)\) and write an expression for \(c_p\).

Invert the order of sum and integral,

\[ \int_0^{2\pi} dx e^{-ipx} f(x) = \int_0^{2\pi} dx \sum_{m=-\infty}^{\infty} C_m e^{i(m-p)x} = \sum_{m=-\infty}^{\infty} C_m \int_0^{2\pi} dx e^{i(m-p)x}. \]

For \(p \neq m\), since \(m-p\) is a (nonzero) integer, the integral vanishes

\[ \int_0^{2\pi} dx e^{i(m-p)x} = \frac{1}{i(m-p)} \left[ e^{2\pi i(m-p)} - 1 \right] = 0, \]

while for \(m = p\) the integral is just \(\int dx (1) = 2\pi\), and

\[ \sum_{m=-\infty}^{\infty} C_m \int_0^{2\pi} dx e^{i(m-p)x} = \sum_{m=-\infty}^{\infty} C_m 2\pi \delta_{mp} = 2\pi C_p. \]

Thus,

\[ C_p = \frac{1}{2\pi} \int_0^{2\pi} dx e^{-ipx} f(x). \]

(b) Compute the Fourier coefficients \(c_m\) for the Dirac \(\delta(x)\) (plus periodic repetitions).

From (a),

\[ c_m = \frac{1}{2\pi} \int dx e^{-imx} \delta(x) = \frac{1}{2\pi} e^{-imx} \bigg|_{x=0} = \frac{1}{2\pi}. \]

Yes, that’s it.
(c) Compute the partial sum $\delta_M$ of the series including terms up to order $M$. Sketch a plot for some moderate value of $M$.

$$\sum_{m=-M}^{M} \frac{1}{2\pi} e^{imx} = \frac{e^{-iMx}}{2\pi} \sum_{m=0}^{2M} (e^{ix})^m = \frac{e^{-iMx}}{2\pi} \frac{1 - e^{i(2M+1)x}}{1 - e^{ix}}$$

$$= \frac{1}{2\pi} \frac{e^{i(M+1)x} - e^{-iMx}}{e^{ix} - 1} = \frac{1}{2\pi} \frac{e^{i(M+\frac{1}{2})x} - e^{-i(M+\frac{1}{2})x}}{e^{i(\frac{1}{2}x)} - e^{-i(\frac{1}{2}x)}} = \frac{\sin[(M + \frac{1}{2})x]}{2\pi \sin(\frac{1}{2}x)}.$$  

The plot shows this function for $M = 32$.

(Note that this problem is different from the example considered in class.)