This is a closed-book exam. If you have any questions about what is asked, please ask earlier rather than later.

Some possible useful information:

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n
\]

\[-\ln(1-x) = x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + \cdots = \sum_{n=1}^{\infty} \frac{1}{n} x^n
\]

\[(1 + x)^p = 1 + p x + \frac{1}{2} p(p-1) x^2 + \frac{1}{6} p(p-1)(p-2) x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n! (p-n)!} x^n
\]

\[e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n
\]

\[\sin x = x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \frac{1}{5040} x^7 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}
\]

\[\cos x = 1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 - \frac{1}{720} x^6 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}
\]
1. Let \( a_n = \frac{(3n)!}{(3n)^{3n}} \). Let \( b_n = \frac{(3n)!}{(n!)^3} \).

(a) Which, if either, of the series \( \sum_{n=0}^{\infty} a_n \) or \( \sum_{n=0}^{\infty} b_n \) converges? Why or why not?

Look at the ratio \( \frac{a_{n+1}}{a_n} \):
\[
\frac{a_{n+1}}{a_n} = \frac{(3n + 3)!(3n + 3)^{3n+3}}{(3n)!/(3n)^{3n}} = \frac{(3n + 3)!}{(3n)!} \left( \frac{n + 1}{n} \right)^{3n}
\]

The first factor gives 1 in the limit \( n \to \infty \), and the \( 3n \)th power gives \( 1/e^3 \), as seen from its logarithm:
\[
3n \ln\left( \frac{n}{n+1} \right) = 3n \ln\left( \frac{1}{1+1/n} \right) = -3n \ln(1 + \frac{1}{n}) = -3n \left( \frac{1}{n} + \frac{1}{2n^2} + \cdots \right) = -3 + O\left( \frac{1}{n} \right) \to -3.
\]

Thus,
\[
\frac{a_{n+1}}{a_n} \to \frac{1}{e^3} < 1,
\]
and, by the ratio test or comparison with the geometric series, this series converges.

Look at the ratio \( \frac{b_{n+1}}{b_n} \):
\[
\frac{b_{n+1}}{b_n} = \frac{(3n + 3)!(n+1)!^3}{(3n)!/(n!)^3} = \frac{(3n + 3)!}{(3n)!} \left[ \frac{n!}{(n+1)!} \right]^3 = \frac{(3n + 3)(3n + 2)(3n + 1)}{(n+1)^3}
\]

Thus,
\[
\frac{b_{n+1}}{b_n} \to 3^3 > 1,
\]
and by the ratio test this series diverges.

(b) For the series in part (a) that has the worst divergence, or the slowest convergence (or, if they behave similarly, pick one), does the power series \( \sum c_n x^n \) \( c_n = a_n \) or \( b_n \) converge for some range of values of \( x \)? If not, explain why; if yes, find the interval of convergence.

The work here is already done: The sum \( \sum b_n x^n \) will converge for \( |x| < 1/27 \).

(c) For the series in part (a) that has the best convergence, or the slowest divergence (or, if they behave similarly, pick the other one), does the power series \( \sum c_n x^n \) \( c_n = a_n \) or \( b_n \) converge for some range of values of \( x \)? If not, explain why; if yes, find the interval of convergence.

And the work here is also already done: The sum \( \sum a_n x^n \) will converge for \( |x| < e^3 \approx 20.0855 \).
2. The logarithmic probability distribution has probabilities \( p_n = C \frac{p^n}{n} \), \( n = 1, 2, \ldots \), that a discrete random variable takes the value \( n \), where \( p \) is a constant, \( 0 < p < 1 \).

(a) What is the value of \( C \) so that \( \sum_{n=1}^{\infty} p_n = 1 \)?

Using a formula from the cover sheet,

\[
\sum_{n=1}^{\infty} C \frac{p^n}{n} = C \left[ -\ln(1 - p) \right] = 1, \quad C = \frac{1}{-\ln(1 - p)}. \]

(b) Compute the average or expected value of \( n \), \( \langle n \rangle = \sum n p_n \). Compute the “second factorial moment”, \( \langle n(n - 1) \rangle \).

Using another formula from the cover sheet,

\[
\langle n \rangle = \sum_{n=1}^{\infty} n p_n = C \sum_{n=1}^{\infty} p^n = C p \sum_{n=1}^{\infty} p^{n-1} = \frac{Cp}{1-p} = -\frac{p}{(1-p) \ln(1-p)}. \]

There is no formula on the cover sheet that has \( n(n - 1) \), but

\[
\langle n \rangle = \sum_{n=1}^{\infty} n(n - 1) p_n = C \sum_{n=1}^{\infty} (n - 1) p^n = C p^2 \sum_{n=1}^{\infty} (n - 1) p^{n-2} = C p^2 \sum_{n=1}^{\infty} \frac{dp}{dp} p^{n-1} = C p^2 \frac{d}{dp} \frac{1}{1-p} = -\frac{p^2}{(1-p)^2 \ln(1-p)}. \]

(c) The generating function \( G(z) \) is defined as \( G(z) = \sum_{k=1}^{\infty} p_k z^k \). Find \( G(z) \) for the logarithmic distribution.

The sum is the same as in (a),

\[
\sum_{k=1}^{\infty} p_k z^k = \sum_{k=1}^{\infty} C \frac{p^k}{k} z^k = C \sum_{k=1}^{\infty} \frac{1}{k} (pz)^k = -\frac{\ln(1 - pz)}{-\ln(1 - p)}. \]
(d) Show that \( \frac{d^n G(z)}{dz^n} \bigg|_{z=0} \) is related to \( p_n \). Show that the moments \( \langle n \rangle \) and \( \langle n(n-1) \rangle \) of part (b) are related to derivatives of \( G(z) \) evaluated at \( z = 1 \). Use your \( G(z) \) to find these moments. Do you get the same result?

Derivatives of \( z^k \) bring factors of \( k, k-1 \), etc., and reduce the power of \( z \); \( n \) derivatives of \( z^k \) vanishes if \( n > k \), is equal to \( n! \) when \( n = k \), and is proportional to \( z^{k-n} \), a positive power, when \( n < k \). Thus, in the sum over \( k \), all terms with \( k < n \) give zero (too many derivatives), all terms with \( k > n \) give zero when evaluated at \( z = 0 \) (positive powers of \( z \), and only the term \( k = n \) survives:

\[
\left. \frac{d^n G(z)}{dz^n} \right|_{z=0} = n! \, p_n. 
\]

Similarly,

\[
\frac{dG(z)}{dz} = \sum_{k=1}^{\infty} p_k k \, z^{p-1}, \quad \frac{d^2 G(z)}{dz^2} = \sum_{k=1}^{\infty} p_k k(k-1) \, z^{k-2},
\]

and so evaluated at \( z = 1 \)

\[
\left. \frac{dG(z)}{dz} \right|_{z=1} = \sum_{n=1}^{\infty} n \, p_n = \langle n \rangle, \quad \left. \frac{d^2 G(z)}{dz^2} \right|_{z=1} = \sum_{n=1}^{\infty} n(n-1) \, p_n = \langle n(n-1) \rangle. 
\]

For \( G = \ln(1 - pz)/\ln(1 - p) \),

\[
\frac{dG}{dz} = -\frac{p}{(1 - pz) \ln(1 - p)}, \quad \frac{d^2 G}{dz^2} = -\frac{p^2}{(1 - pz)^2 \ln(1 - p)},
\]

and evaluated at \( z = 1 \) give

\[
\langle n \rangle = -\frac{p}{(1 - p) \ln(1 - p)}, \quad \langle n(n-1) \rangle = -\frac{p^2}{(1 - p)^2 \ln(1 - p)}. 
\]
3. The function \( j(x) \) is given by \( j(x) = [(3 - x^2) \sin x - 3x \cos x]/x^3 \).

How does \( j(x) \) behave for small \( x \)? (Find at least the first nontrivial term of an expansion of \( j(x) \) for \( x \) near 0). How does \( j(x) \) behave for large \( x \)? Sketch a plot of \( j(x) \).

Be sure to take enough terms, up to \( x^5 \) in \( \sin x \) and up to \( x^4 \) in \( \cos x \). The numerator is then

\[
x^3 j \approx (3 - x^2) (x - \frac{1}{6} x^3 + \frac{1}{120} x^5 + \cdots) - 3x (1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 + \cdots)
\]

\[
= 3x - \frac{1}{2} x^3 + \frac{1}{40} x^5 - x^3 + \frac{1}{6} x^5 - 3x + \frac{3}{2} x^4 - \frac{1}{8} x^5 + \cdots
\]

The coefficients of \( x \) sum to zero as \( 3 - 3 \); the coefficients of \( x^3 \) sum to zero as \( -\frac{1}{2} - 1 + \frac{3}{2} \); and the coefficient of \( x^5 \) is \( \frac{1}{40} + \frac{1}{6} - \frac{1}{8} = \frac{3}{120} + \frac{20}{120} - \frac{15}{120} = \frac{8}{120} = \frac{1}{15} \), and so for small \( x \),

\[
j(x) \approx \frac{1}{15} x^2 + \cdots.
\]

For large \( x \),

\[
j(x) \approx -\frac{x^2 \sin x}{x^3} = -\frac{\sin x}{x}.
\]
4. Use the method of Lagrange multipliers to find the point on the circle \( x^2 + y^2 = 1 \) that is closest to the point \((1, -\frac{3}{4})\).

The squared distance is \( d^2 = (x - 1)^2 + (y + \frac{3}{4})^2 \), the constraint is \( \phi = x^2 + y^2 - 1 = 0 \), and the method of Lagrange multipliers says find extrema of

\[
F = (x - 1)^2 + (y + \frac{3}{4})^2 + \lambda(x^2 + y^2 - 1).
\]

Extrema are where the derivatives vanish:

\[
\begin{align*}
\frac{\partial F}{\partial x} &= 2(x - 1) + 2\lambda x = 0, \\
\frac{\partial F}{\partial y} &= 2(y + \frac{3}{4}) + 2\lambda y = 0, \\
\frac{\partial F}{\partial \lambda} &= x^2 + y^2 - 1 = 0
\end{align*}
\]

The first two of these say

\[
x = \frac{1}{1 + \lambda}, \quad y = \frac{-3/4}{1 + \lambda},
\]

and in the third these give

\[
\frac{1}{(1 + \lambda)^2} \left(1 + \frac{9}{16}\right) = \frac{1}{(1 + \lambda)^2} \frac{25}{16} = 1, \quad \frac{1}{(1 + \lambda)} = \pm \frac{4}{5}.
\]

This then gives the positions of the extrema

\[
x = \pm \frac{4}{5}, \quad y = \pm \frac{3}{5}.
\]

Clearly, the first sign choice is the closest (for instance, in the same quadrant), and the other is is the furthest (in the opposite quadrant). A plot of the geometry appears at the end.
Bonus: Use the method of Lagrange multipliers to find the point on the circle \( x^2 + y^2 = 1 \) that is closest to the line \( 3x - 4y = 6 \).

This time we need to find a point \((x, y)\) on the circle and also another point, call it \((x', y')\), on the line,

\[
F = (x - x')^2 + (y - y')^2 + \lambda_1(x^2 + y^2 - 1) + \lambda_2(3x' - 4y' - 6).
\]

The extrema occur at

\[
\frac{\partial F}{\partial x} = 2(x - x') + 2\lambda_1 x = 0, \quad \frac{\partial F}{\partial y} = 2(y - y') + 2\lambda_1 y = 0, \\
\frac{\partial F}{\partial x'} = 2(x' - x) + 3\lambda_2 = 0, \quad \frac{\partial F}{\partial y'} = 2(y' - y) - 4\lambda_2 = 0, \\
\frac{\partial F}{\partial \lambda_1} = x^2 + y^2 - 1 = 0, \quad \frac{\partial F}{\partial \lambda_2} = 3x' - 4y' - 6 = 0.
\]

This is six equations in six unknowns. The middle pair can be solved for

\[
x - x' = \frac{3\lambda_2}{2}, \quad y - y' = -2\lambda_2;
\]

with this information the first pair give

\[
x = -\frac{3\lambda_2}{2\lambda_1}, \quad y = \frac{2\lambda_2}{\lambda_1};
\]

and in turn the fifth equation gives

\[
x^2 + y^2 = \frac{9\lambda_2^2}{4\lambda_1^2} + \frac{4\lambda_2^2}{\lambda_1^2} = \frac{25\lambda_2^2}{4\lambda_1^2} = 1, \quad \frac{\lambda_2}{\lambda_1} = \pm \frac{2}{5}.
\]

Thus,

\[
x = \pm \frac{3}{5}, \quad y = \mp \frac{4}{5}.
\]

Once again, + is the nearest, – is the furthest. This works as long as \(\lambda_1 \neq 0\); if \(\lambda_1 = 0\), then \(x = x', y = y', \lambda_2 = 0\), and \(x, y\) must solve both of

\[
x^2 + y^2 = 1, \quad 3x - 4y = 6,
\]

but this pair has only complex solutions.
The blue points are at extremal distance (nearest and furthest) from the point \((1, -\frac{3}{4})\), which happens to be on the same line as in the Bonus. The red points extremize distance from the line to the circle.