PHZ 3113 Fall 2017

Exam 1, Friday September 22

Name: _____

This is a closed-book exam. If you have any questions about what is asked, please ask earlier rather than later.

Some possible useful information:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$-\ln(1-x) = x + \frac{1}{2}\,x^2 + \frac{1}{3}\,x^3 + \frac{1}{4}\,x^4 + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}\,x^n$$

$$(1+x)^p = 1 + p\,x + \frac{1}{2}\,p(p-1)\,x^2 + \frac{1}{6}\,p(p-1)(p-2)\,x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\,\frac{p!}{(p-n)!}\,x^n$$

$$e^x = 1 + x + \frac{1}{2}\,x^2 + \frac{1}{6}\,x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\,x^n$$

$$\sin x = x - \frac{1}{6}\,x^3 + \frac{1}{120}\,x^5 - \frac{1}{5040}\,x^7 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\,x^{2k+1}$$

$$\cos x = 1 - \frac{1}{2}\,x^2 + \frac{1}{24}\,x^4 - \frac{1}{720}\,x^6 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\,x^{2k}$$

1. *Let* $a_n = \dfrac{(3n)!}{(3n)^{3n}}$. *Let* $b_n = \dfrac{(3n)!}{(n!)^3}$.

(*a*) *Which, if either, of the series* $\sum_{n=0}^{\infty} a_n$ *or* $\sum_{n=0}^{\infty} b_n$ *converges? Why or why not?*

Look at the ratio a_{n+1}/a_n:

$$\frac{a_{n+1}}{a_n} = \frac{(3n+3)!/(3n+3)^{3n+3}}{(3n)!/(3n)^{3n}} = \frac{(3n+3)!}{(3n)!}\,\frac{(3n)^{3n}}{(3n+3)^{3n+3}}$$

$$= \frac{(3n+3)(3n+2)(3n+1)}{(3n+3)^3}\left(\frac{n}{n+1}\right)^{3n}.$$

The first factor gives 1 in the limit $n \to \infty$, and the $3n$th power gives $1/e^3$, as seen from its logarithm:

$$3n\,\ln(\frac{n}{n+1}) = 3n\,\ln(\frac{1}{1+1/n}) = -3n\,\ln(1+\frac{1}{n}) = -3n\,(\frac{1}{n}+\frac{1}{2n^2}+\cdots) = -3+\mathcal{O}(\frac{1}{n}) \to -3.$$

Thus,

$$\frac{a_{n+1}}{a_n} \to \frac{1}{e^3} < 1,$$

and, by the ratio test or comparison with the geometric series, this series converges.

Look at the ratio b_{n+1}/b_n:

$$\frac{b_{n+1}}{b_n} = \frac{(3n+3)!/[(n+1)!]^3}{(3n)!/(n!)^3} = \frac{(3n+3)!}{(3n)!}\left[\frac{n!}{(n+1)!}\right]^3 = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3}$$

Thus,

$$\frac{b_{n+1}}{b_n} \to 3^3 > 1,$$

and by the ratio test this series diverges.

(*b*) *For the series in part (a) that has the worst divergence, or the slowest convergence (or, if they behave similarly, pick one), does the power series* $\sum c_n\,x^n$ *(*$c_n = a_n$ *or* b_n*) converge for some range of values of* x*? If not, explain why; if yes, find the interval of convergence.*

The work here is already done: The sum $\sum b_n x^n$ will converge for $|x| < 1/27$.

(*c*) *For the series in part (a) that has the best convergence, or the slowest divergence (or, if they behave similarly, pick the other one), does the power series* $\sum c_n\,x^n$ *(*$c_n = a_n$ *or* b_n*) converge for some range of values of* x*? If not, explain why; if yes, find the interval of convergence.*

And the work here is also already done: The sum $\sum a_n x^n$ will converge for $|x| < e^3 \approx 20.0855$.

2. *The* logarithmic probability distribution *has probabilities* $p_n = C\,\dfrac{p^n}{n}$, $n = 1,\ 2,\ \ldots,$ *that a discrete random variable takes the value* n, *where* p *is a constant,* $0 < p < 1$.

(a) *What is the value of* C *so that* $\displaystyle\sum_{n=1}^{\infty} p_n = 1$?

Using a formula from the cover sheet,

$$\sum_{n=1}^{\infty} C\,\frac{p^n}{n} = C\,[-\ln(1-p)] = 1, \qquad C = \frac{1}{-\ln(1-p)}. \qquad \blacksquare$$

(b) *Compute the average or expected value of* n, $\langle\, n \,\rangle = \sum n\, p_n$. *Compute the "second factorial moment",* $\langle\, n(n-1) \,\rangle$.

Using another formula from the cover sheet,

$$\langle\, n \,\rangle = \sum_{n=1}^{\infty} n\, p_n = C \sum_{n=1}^{\infty} p^n = Cp \sum_{n=1}^{\infty} p^{n-1} = \frac{Cp}{1-p} = -\frac{p}{(1-p)\,\ln(1-p)}. \qquad \blacksquare$$

There is no formula on the cover sheet that has $n(n-1)$, but

$$\begin{aligned}\langle\, n \,\rangle &= \sum_{n=1}^{\infty} n(n-1)\, p_n = C \sum_{n=1}^{\infty} (n-1)\, p^n = Cp^2 \sum_{n=1}^{\infty} (n-1)\, p^{n-2} = Cp^2 \sum_{n=1}^{\infty} \frac{d}{dp}\, p^{n-1} \\ &= Cp^2\, \frac{d}{dp} \sum_{n=1}^{\infty} p^{n-1} = Cp^2 \frac{d}{dp} \frac{1}{1-p} = -\frac{p^2}{(1-p)^2\,\ln(1-p)}. \qquad \blacksquare\end{aligned}$$

(c) *The generating function* $G(z)$ *is defined as* $G(z) = \displaystyle\sum_{k=1}^{\infty} p_k\, z^k$. *Find* $G(z)$ *for the logarithmic distribution.*

The sum is the same as in (a),

$$\sum_{k=1}^{\infty} p_k\, z^k = \sum_{k=1}^{\infty} C\,\frac{p^k}{k}\, z^k = C \sum_{k=1}^{\infty} \frac{1}{k}\,(pz)^k = \frac{-\ln(1-pz)}{-\ln(1-p)}. \qquad \blacksquare$$

(d) Show that $\left.\dfrac{d^n G(z)}{dz^n}\right|_{z=0}$ *is related to* p_n*. Show that the moments* $\langle\, n \,\rangle$ *and* $\langle\, n(n-1) \,\rangle$ *of part (b) are related to derivatives of* $G(z)$ *evaluated at* $z = 1$*. Use your* $G(z)$ *to find these moments. Do you get the same result?*

Derivatives of z^k bring factors of k, $k-1$, etc., and reduce the power of z; n derivatives of z^k vanishes if $n > k$, is equal to $n!$ when $n = k$, and is proportional to z^{k-n}, a positive power, when $n < k$. Thus, in the sum over k, all terms with $k < n$ give zero (too many derivatives), all terms with $k > n$ give zero when evaluated at $z = 0$ (positive powers of z, and only the term $k = n$ survives:

$$\left.\frac{d^n G(z)}{dz^n}\right|_{z=0} = n!\, p_n. \qquad \blacksquare$$

Similarly,

$$\frac{dG(z)}{dz} = \sum_{k=1}^{\infty} p_k\, k\, z^{p-1}, \qquad \frac{d^2G(z)}{dz^2} = \sum_{k=1}^{\infty} p_k\, k(k-1)\, z^{k-2},$$

and so evaluated at $z = 1$

$$\left.\frac{dG(z)}{dz}\right|_{z=1} = \sum_{n=1}^{\infty} n\, p_n = \langle\, n \,\rangle, \qquad \left.\frac{d^2G(z)}{dz^2}\right|_{z=1} = \sum_{n=1}^{\infty} n(n-1)\, p_n = \langle\, n(n-1) \,\rangle\,. \qquad \blacksquare$$

For $G = \ln(1-pz)/\ln(1-p)$,

$$\frac{dG}{dz} = -\frac{p}{(1-pz)\ln(1-p)}, \qquad \frac{d^2G}{dz^2} = -\frac{p^2}{(1-pz)^2\ln(1-p)},$$

and evaluated at $z = 1$ give

$$\langle\, n \,\rangle = -\frac{p}{(1-p)\ln(1-p)}, \qquad \langle\, n(n-1) \,\rangle = -\frac{p^2}{(1-p)^2\ln(1-p)}. \qquad \blacksquare$$

3. *The function $j(x)$ is given by $j(x) = [(3 - x^2)\sin x - 3x\cos x]/x^3$.*

How does $j(x)$ behave for small x? (Find at least the first nontrivial term of an expansion of $j(x)$ for x near 0). How does $j(x)$ behave for large x? Sketch a plot of $j(x)$.

Be sure to take enough terms, up to x^5 in $\sin x$ and up to x^4 in $\cos x$. The numerator is then

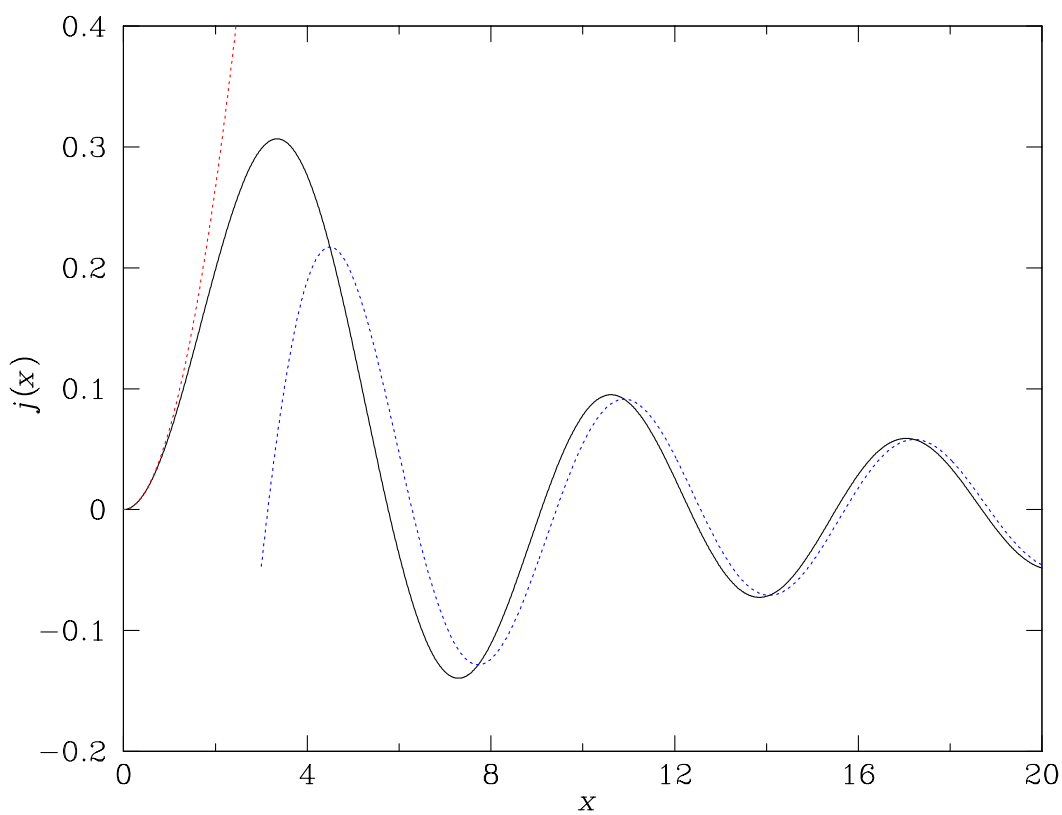
$$\begin{aligned} x^3 j &\approx (3 - x^2)\,(x - \tfrac{1}{6}\,x^3 + \tfrac{1}{120}\,x^5 + \cdots) - 3x\,(1 - \tfrac{1}{2}\,x^2 + \tfrac{1}{24}\,x^4 + \cdots) \\ &= 3x - \tfrac{1}{2}\,x^3 + \tfrac{1}{40}\,x^5 - x^3 + \tfrac{1}{6}\,x^5 - 3x + \tfrac{3}{2}\,x^4 - \tfrac{1}{8}\,x^5 + \cdots \end{aligned}$$

The coefficients of x sum to zero as $3 - 3$; the coefficients of x^3 sum to zero as $-\frac{1}{2} - 1 + \frac{3}{2}$; and the coefficient of x^5 is $\frac{1}{40} + \frac{1}{6} - \frac{1}{8} = \frac{3}{120} + \frac{20}{120} - \frac{15}{120} = \frac{8}{120} = \frac{1}{15}$, and so for small x,

$$j(x) \approx \frac{1}{15}\,x^2 + \cdots. \qquad \blacksquare$$

For large x,

$$j(x) \approx -\frac{x^2 \sin x}{x^3} = -\frac{\sin x}{x}. \qquad \blacksquare$$



4. *Use the method of Lagrange multipliers to find the point on the circle* $x^2 + y^2 = 1$ *that is closest to the point* $(1, -\frac{3}{4})$.

The squared distance is $d^2 = (x-1)^2 + (y+\frac{3}{4})^2$, the constraint is $\phi = x^2 + y^2 - 1 = 0$, and the method of Lagrange multipliers says find extrema of

$$F = (x-1)^2 + (y+\tfrac{3}{4})^2 + \lambda(x^2 + y^2 - 1).$$

Extrema are where the derivatives vanish:

$$\begin{aligned}
\frac{\partial F}{\partial x} &= 2(x-1) + 2\lambda x = 0, \\
\frac{\partial F}{\partial y} &= 2(y+\tfrac{3}{4}) + 2\lambda y = 0, \\
\frac{\partial F}{\partial \lambda} &= x^2 + y^2 - 1 = 0
\end{aligned}$$

The first two of these say

$$x = \frac{1}{1+\lambda}, \qquad y = \frac{-3/4}{1+\lambda},$$

and in the third these give

$$\frac{1}{(1+\lambda)^2}\left(1 + \frac{9}{16}\right) = \frac{1}{(1+\lambda)^2}\,\frac{25}{16} = 1, \qquad \frac{1}{(1+\lambda)} = \pm\frac{4}{5}.$$

This then gives the positions of the extrema

$$x = \pm\frac{4}{5}, \qquad y = \mp\frac{3}{5}. \qquad \blacksquare$$

Clearly, the first sign choice is the closest (for instance, in the same quadrant), and the other is is the furthest (in the opposite quadrant). A plot of the geometry appears at the end.

Bonus: *Use the method of Lagrange multipliers to find the point on the circle* $x^2 + y^2 = 1$ *that is closest to the line* $3x - 4y = 6$.

This time we need to find a point (x, y) on the circle and also another point, call it (x', y'), on the line,

$$F = (x - x')^2 + (y - y')^2 + \lambda_1(x^2 + y^2 - 1) + \lambda_2(3x' - 4y' - 6).$$

The extrema occur at

$$\frac{\partial F}{\partial x} = 2(x - x') + 2\lambda_1 x = 0, \qquad \frac{\partial F}{\partial y} = 2(y - y') + 2\lambda_1 y = 0,$$

$$\frac{\partial F}{\partial x'} = 2(x' - x) + 3\lambda_2 = 0, \qquad \frac{\partial F}{\partial y'} = 2(y' - y) - 4\lambda_2 = 0,$$

$$\frac{\partial F}{\partial \lambda_1} = x^2 + y^2 - 1 = 0, \qquad \frac{\partial F}{\partial \lambda_2} = 3x' - 4y' - 6 = 0.$$

This is six equations in six unknowns. The middle pair can be solved for

$$x - x' = \frac{3\lambda_2}{2}, \qquad y - y' = -2\lambda_2;$$

with this information the first pair give

$$x = -\frac{3\lambda_2}{2\lambda_1}, \qquad y = +\frac{2\lambda_2}{\lambda_1};$$

and in turn the fifth equation gives

$$x^2 + y^2 = \frac{9\,\lambda_2^2}{4\,\lambda_1^2} + \frac{4\,\lambda_2^2}{\lambda_1^2} = \frac{25\,\lambda_2^2}{4\,\lambda_1^2} = 1, \qquad \frac{\lambda_2}{\lambda_1} = \pm\frac{2}{5}.$$

Thus,

$$x = \pm\,\frac{3}{5}, \qquad y = \mp\,\frac{4}{5}.$$

Once again, $+$ is the nearest, $-$ is the furthest. This works as long as $\lambda_1 \neq 0$; if $\lambda_1 = 0$, then $x = x'$, $y = y'$, $\lambda_2 = 0$, and x, y must solve both of

$$x^2 + y^2 = 1, \qquad 3x - 4y = 6,$$

but this pair has only complex solutions.

The blue points are at extremal distance (nearest and furthest) from the point $(1, -\frac{3}{4})$, which happens to be on the same line as in the Bonus. The red points extremize distance from the line to the circle.