PHZ 3113 Fall 2017

Homework #4, Due Wednesday, October 4

1. Define elliptical coordinates u, v by $x = \sqrt{u^2 + c^2}\,\cos v, \quad y = u\,\sin v\,.$

(a) Write the area element $dA = dx\,dy$ in terms of elliptical u, v.

(b) Using elliptical coordinates, compute the area of an ellipse with semimajor axis a, semiminor axis b, where $a^2 = b^2 + c^2$.

2. Let the scalar field f be given by $f(x, y) = \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}}$.

(a) Compute the components of the gradient ∇f. Where is the derivative troublesome?

(b) Compute the directional derivative of $\nabla_{\boldsymbol{u}} f$ in the direction $\boldsymbol{u} = \hat{\boldsymbol{x}}\,\cos\alpha + \hat{\boldsymbol{y}}\sin\alpha$. Write your answer in terms of polar coordinates ρ, ϕ, where $x = \rho\cos\phi$, $y = \rho\sin\phi$. What is the directional derivative along a radial ray (fixed ϕ) as $\rho \to 0$?

3. Let the vector field $\boldsymbol{v}$ be

$$\boldsymbol{v} = 2xy\,\hat{\boldsymbol{x}} + (x^2 + y^2)\,\hat{\boldsymbol{y}}\,.$$

(a) Compute $\nabla \cdot \boldsymbol{v}$. Compute $\nabla \times \boldsymbol{v}$.

(b) Compute $\nabla(\nabla \cdot \boldsymbol{v})$ and $\nabla^2 \boldsymbol{v}$. Compute $\nabla \times (\nabla \times \boldsymbol{v})$ two different ways.

4. The electric and gravitational forces between point particles are both inverse square, and in analogy with electrostatics, the gravitational field $\boldsymbol{g}$ can be obtained from a gravitational potential $\boldsymbol{g} = -\nabla\Phi$ that satisfies $\nabla^2\Phi = 4\pi G\rho$. In numerical simulations of large-scale cosmological structure, it is sometimes useful to "soften" the Newtonian gravitational potential $\Phi = -GM/r$ of a point particle of mass M as

$$\Phi_a = -\frac{GM}{\sqrt{r^2 + a^2}}.$$

(a) What is the gravitational field $\boldsymbol{g}$ derived from Φ_a? How does $\boldsymbol{g}$ behave as $r \to \infty$? Show that the softening introduces a maximum in the gravitational force between particles.

(b) The softened Φ_a can be interpreted as the potential of a slightly fuzzy particle with a mass density $\rho_a(r)$. What is $\rho_a(r)$? What is the total mass, $M_a = \int d^3v\,\rho_a$? What radius $r_{1/2}$ contains half the mass?