PHZ 3113 Fall 2017

Homework #7, Due Friday, November 10

1. Find the eigenvalues and eigenvectors of the matrices

$$A = \begin{pmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}.$$

What is the angle between the eigenvectors in each case?

2. If the matrix M depends on a parameter λ, show that $\dfrac{d}{d\lambda} M^{-1} = -M^{-1} \dfrac{dM}{d\lambda} M^{-1}$.

3. Three beads of equal mass m are constrained to lie on a circular hoop. The beads are connected by identical springs of spring constant k. The equations of motion for displacements θ_k are

$$\begin{aligned} m\,\ddot{\theta}_1 &= k\,(\theta_2 + \theta_3 - 2\theta_1), \\ m\,\ddot{\theta}_2 &= k\,(\theta_3 + \theta_1 - 2\theta_2), \\ m\,\ddot{\theta}_3 &= k\,(\theta_1 + \theta_2 - 2\theta_3). \end{aligned}$$

Assume there are oscillatory solutions, $\theta_1(t) = \theta_1 \cos(\omega t)$, etc. What are the characteristic frequencies of oscillations? What are the oscillation patterns? Because of the high degree of symmetry of the problem, you may be able to guess some of the characteristic modes, but show that you can choose modes that are orthogonal.

4. Let $Q = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Compute the powers Q^2, Q^3, Q^4. What is Q to any odd power, Q^{2k+1}. What is Q to any even power, Q^{2k}. What is $\exp(\zeta Q)$?

5. Let the three matrices

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form a vector $\boldsymbol{J} = (J_1, J_2, J_3)$.

(a) The *commutator* of two matrices is defined to be $[A, B] = AB - BA$. Compute the commutators $[J_i, J_j]$. There are nine ij combinations, but only three need to be multiplied out in detail. (Why is that?) See if you can encapsulate your results using ϵ_{ijk}.

(b) The action of a rotation by $\boldsymbol{\theta} = \theta\,\hat{\boldsymbol{n}}$ (rotation by angle θ about axis $\hat{\boldsymbol{n}}$) on a vector $\boldsymbol{v}$ is $\boldsymbol{v}' = R\boldsymbol{v}$, where $R = \exp(\boldsymbol{\theta}\cdot\boldsymbol{J})$. Show that for rotation by a small angle $\Delta\boldsymbol{\theta}$, the change in $\boldsymbol{v}$ is $\Delta\boldsymbol{v} = \Delta\boldsymbol{\theta}\times\boldsymbol{v}$.

(c) The action of a rotation on a matrix is by conjugation, $M' = RMR^T$, and since for each component $J_i^T = -J_i$, this can be written $M' = \exp(\boldsymbol{\theta}\cdot\boldsymbol{J})\,M\,\exp(-\boldsymbol{\theta}\cdot\boldsymbol{J})$. For rotation by small angle $\Delta\boldsymbol{\theta}$, acting on $\boldsymbol{J}$ what is the change in $\boldsymbol{J}$? (A change in $\boldsymbol{J}$ means a change in axis direction; confuse yourself by reading about "active" and "passive" rotations.)