1. Let the differential operator $\mathcal{D}f = \frac{d}{dx} \left(1 - x^2\right) \frac{df}{dx}$ act on functions defined on the interval $(-1, 1)$.

   (a) Show that for any functions $f(x)$, $g(x)$, \[\int_{-1}^{1} dx \int_{-1}^{1} dx \int_{-1}^{1} \mathcal{D}g(x) f(x) dx f(x) dx f(x) \approx \int_{-1}^{1} dx \mathcal{D}f(x) g(x);\] that is, that $\mathcal{D}$ is self-adjoint.

   (b) Suppose $P_m$ and $P_n$ are eigenfunctions of $\mathcal{D}$, such that $\mathcal{D}P_m(x) + m(m + 1) P_m(x) = 0$ and $\mathcal{D}P_n(x) + n(n + 1) P_n(x) = 0$. Show that for $n \neq m$, $P_m(x)$ and $P_n(x)$ are orthogonal on $(-1, 1)$.

   (c) Verify that $P_0 = 1$ and $P_1 = x$ are eigenfunctions with the suggested eigenvalues.

   (d) Let $P_2(x)$ be a polynomial of degree 2, $P_2 = c_0 + c_1 x + c_2 x^2$. Choose $c_0$ and $c_1$ so that $P_2$ is orthogonal to $P_0$ and $P_1$. Choose $c_2$ so that $P_2(1) = 1$. Write your resulting $P_2(x)$. Is your $P_2(x)$ an eigenfunction of $\mathcal{D}P_2 + p(p + 1) P_2$ for some eigenvalue $p$?

2. Let $f(x)$ be the triangular function that takes on the value $f = 0$ for $|x| > a$, joined linearly from the points at $x = \pm a$ to $f = 1/a$ at $x = 0$ (see plot on next page).

   (a) Compute the Fourier transform $\tilde{f}(k)$.

   (b) What does $\tilde{f}(k)$ become as $a \to 0$? What does $f(x)$ become as $a \to 0$?

   (c) Compute the value of the integral $I_1 = \int_{-\infty}^{\infty} \frac{1 - \cos y}{y^2} dy$.

   (d) Compute the value of the integral $I_2 = \int_{-\infty}^{\infty} \frac{(1 - \cos y)^2}{y^4} dy$. 

![Plot of triangular function](image-url)