Legendre Polynomials: Rodriques' Formula and Recursion Relations

Jackson says "By manipulation of the power series solutions it is possible to obtain a compact representation of the Legendre polynomials known as Rodrigues' formula." Here is a proof that Rodrigues' formula indeed produces a solution to Legendre's differential equation. From the differential equation, assuming a series solution $P_n = \sum a_j x^j$ ($\alpha = 0$) we obtained the relation

$$a_{j+2} = \frac{j(j+1) - n(n+1)}{(j+1)(j+2)}\, a_j$$

[JDJ (3.14), with $\alpha = 0$]. With $j = n - 2k$, this is satisfied by

$$a_{n-2k} = (-1)^k \frac{(2n-2k)!}{2^n\, k!\,(n-k)!\,(n-2k)!},$$

where $1/2^n$ is conventional. So, we can write

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n\, k!\,(n-k)!\,(n-2k)!}\, x^{n-2k},$$

where $[n/2]$ denotes the "greatest integer" or the integer part. For integer $n - 2k$, this is

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k}{2^n k!(n-k)!} \left(\frac{d}{dx}\right)^n x^{2n-2k} = \frac{1}{2^n\, n!} \left(\frac{d}{dx}\right)^n \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\, (-1)^k\, (x^2)^{n-k},$$

where the extra terms introduced by extending the upper limit of the sum from $[n/2]$ to n have zero derivative. By the binomial theorem, this expression is

$$P_n(x) = \frac{1}{2^n\, n!} \left(\frac{d}{dx}\right)^n (x^2 - 1)^n\,. \tag{3.16}$$

Jackson next says, "From Rodrigues' formula it is a straightforward matter" to obtain recursion relations for the P_n. To this end, first prove some relations that are useful in many applications. Let $\mathcal{D} = d/dx$. Then, for any function $f(x)$,

$$\mathcal{D}^l (xf) = x\,(\mathcal{D}^l f) + l\,(\mathcal{D}^{l-1} f).$$

This can be proved by induction: it clearly holds for $l = 0$, for which it reads $xf = xf$, and for $l = 1$ by the product rule for derivatives, $\mathcal{D}(xf) = x(\mathcal{D}f) + f(\mathcal{D}x)$. Suppose it holds for $l - 1$; then

$$\begin{aligned}\mathcal{D}^l (xf) &= \mathcal{D}[\mathcal{D}^{l-1}(xf)] = \mathcal{D}[x(\mathcal{D}^{l-1} f + (l-1)\mathcal{D}^{l-2} f] \\ &= [x(\mathcal{D}^l f) + \mathcal{D}^{l-1} f] + (l-1)\mathcal{D}^{l-1} f = x(\mathcal{D}^l f) + l(\mathcal{D}^{l-1} f).\end{aligned}$$

Apply for $g(x) = xf(x)$:

$$\begin{aligned}\mathcal{D}^l(x^2 f) &= \mathcal{D}^l(x\cdot fx) = x\mathcal{D}^l(fx) + l\mathcal{D}^{l-1}(fx)\\ &= x[x(\mathcal{D}^l f) + l\mathcal{D}^{l-1}f] + l\,[x\mathcal{D}^{l-1}f + (l-1)\mathcal{D}^{l-2}f]\\ &= x^2(\mathcal{D}^l f) + 2lx\,(\mathcal{D}^{l-1}f) + l(l-1)\,(\mathcal{D}^{l-2}f).\end{aligned}$$

This procedure iterated leads to the general and perhaps well known result

$$\mathcal{D}^n(fg) = \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!}\,(\mathcal{D}^k f)(\mathcal{D}^{n-k}g),$$

and so in particular

$$\mathcal{D}^l[(x^2-1)f] = (x^2-1)\,(\mathcal{D}^l f) + 2lx\,(\mathcal{D}^{l-1}f) + l(l-1)\,(\mathcal{D}^{l-2}f). \qquad (*)$$

Now, use $(*)$ to prove the middle of Jackson's (3.29). Apply $\mathcal{D}$ to Rodrigues' formula for P_{l+1}, first commuting $\mathcal{D}^l$ as above and then applying the product rule for derivatives a number of times:

$$\begin{aligned}\mathcal{D}P_{l+1} &= \mathcal{D}\left[\frac{1}{2^{l+1}(l+1)!}\mathcal{D}^{l+1}(x^2-1)^{l+1}\right] = \frac{1}{2(l+1)}\,\mathcal{D}^2\left[\mathcal{D}^l\,\frac{1}{2^l l!}(x^2-1)(x^2-1)^l\right]\\ &= \frac{1}{2(l+1)}\,\frac{1}{2^l l!}\,\mathcal{D}^2\left[(x^2-1)\mathcal{D}^l(x^2-1)^l + 2lx\,\mathcal{D}^{l-1}(x^2-1)^l + l(l-1)\mathcal{D}^{l-2}(x^2-1)^l\right]\\ &= \frac{1}{2(l+1)}\left[\mathcal{D}((x^2-1)\mathcal{D}P_l) + (2P_l + 2x\,\mathcal{D}P_l) + (2lx\,\mathcal{D}P_l + 4l\,P_l) + l(l-1)\,P_l\right].\end{aligned}$$

From Legendre's equation, the first term is $l(l+1)\,P_l$. Gathering terms in $x\,\mathcal{D}P_l$ and P_l,

$$\mathcal{D}P_{l+1} = \frac{1}{2(l+1)}\left[2(l+1)^2 P_l + 2(l+1)x\,\mathcal{D}P_l\right].$$

or, finally, the desired result,

$$\mathcal{D}P_{l+1} = (l+1)P_l + x\,\mathcal{D}P_l. \qquad (3.29b)$$

It seems likely that there is an easier way to get here, but this works.

Here is another one:

$$\begin{aligned}\mathcal{D}P_{l+1} &= \mathcal{D}\Big[\frac{1}{2^{l+1}(l+1)!}\mathcal{D}^{l+1}(x^2-1)^{l+1}\Big] = \frac{1}{2(l+1)}\frac{1}{2^l l!}\,\mathcal{D}^l\left[\mathcal{D}^2(x^2-1)^{l+1}\right]\\ &= \frac{1}{2(l+1)}\frac{1}{2^l l!}\,\mathcal{D}^l\left[4l(l+1)x^2(x^2-1)^{l-1} + 2(l+1)(x^2-1)^l\right]\\ &= \frac{1}{2^l l!}\,\mathcal{D}^l\left[2l(x^2-1+1)(x^2-1)^{l-1} + (x^2-1)^l\right]\\ &= \frac{1}{2^l l!}\,\mathcal{D}^l\left[2l(x^2-1)^{l-1} + (2l+1)(x^2-1)^l\right] = \mathcal{D}P_{l-1} + (2l+1)P_l.\end{aligned}$$

From beginning to end, this says

$$\mathcal{D}P_{l+1} - \mathcal{D}P_{l-1} = (2l+1)P_l\,. \qquad (3.28)$$