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$$\boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \rho(\vec{x}') \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}$$ (1.5)

Coulomb.

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\int d^3x' \frac{\rho(\vec{x}')}{4\pi\varepsilon_0} \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \rho(\vec{x}') \, \vec{\nabla}\cdot\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)$$

Translation of origin doesn't affect derivatives:

$\longrightarrow$ look at $\vec{\nabla}\cdot\left(\frac{\vec{x}}{r^3}\right)$

machinery. $\vec{x} = x_i = \hat{x}_i x_i$ $\quad i = 1,2,3$ (components)

$$r^2 = \vec{x}\cdot\vec{x} = (x_i\hat{x}_i)\cdot(x_j\hat{x}_j)$$

$$= x_i x_j \delta_{ij} = x_j x_j$$ implicit sum

$$\hat{x}_i\cdot\hat{x}_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$ Kronecker delta.

2)

$$\nabla_i x_j = \frac{\partial x_i}{\partial x_j} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} = \delta_{ij}$$

$\underline{11}$ $\quad \frac{\partial x}{\partial x} = 1.$ $\qquad \underline{23}.$ $\frac{\partial y}{\partial z} = 0$

$$\nabla_i r^p = \nabla_i (x_j x_j)^{p/2} = \left(\frac{p}{2}\right)(x_k x_k)^{\frac{p}{2}-1} \left(\nabla_i (x_j x_j)\right)$$

$$= \frac{p}{2}(r)^{p-2}(2\delta_{ij} x_j) = p\, r^{p-2} x_i$$

$\boxed{\delta_{ij} x_j = x_i}$

$$\boxed{\vec{\nabla} r^p = p \vec{x} r^{p-2} = p r^{p-1} \hat{r}}$$

$(p = -3)$

$$\vec{\nabla} \cdot \left(\frac{\vec{x}}{r^3}\right) = \frac{1}{r^3}(\vec{\nabla} \cdot \vec{x}) + \vec{x} \cdot \left(\vec{\nabla} \frac{1}{r^3}\right)$$

$$= \frac{1}{r^3} \cdot \delta_{ij} + \vec{x} \cdot \left(-3 \frac{\vec{x}}{r^5}\right)$$

$$= \frac{3}{r^3} - 3 \frac{\vec{x} \cdot \vec{x}}{r^5} = \underline{\underline{0}}. \checkmark ?$$

(Almost)

(3)

Can't be: point q. $\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$

$$\oint_{\text{sphere}} d^2a\ \vec{E}\cdot\hat{r} = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\cdot 4\pi r^2 = q/\varepsilon_0 .$$

$$= \int_{\text{interior}} d^3x\ \vec{\nabla}\cdot\vec{E} \qquad \left(\begin{matrix}\text{Top Right}\\ \text{Cover. 2}\end{matrix}\right).$$

Problem lies at $r=0$. $\frac{1}{r^3}$ singular

Fix: regulate, look at limit

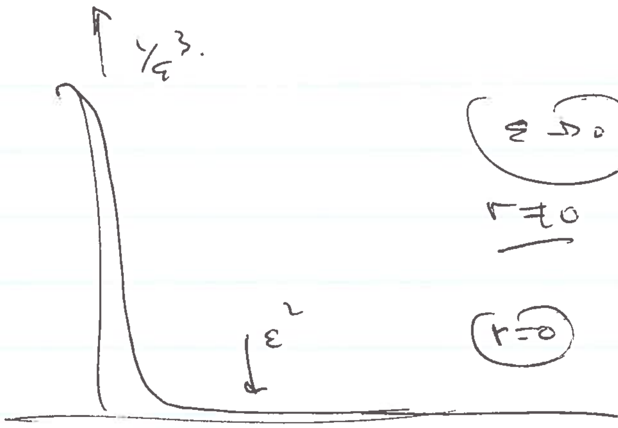
$$\vec{\nabla}\cdot\left(\frac{\vec{x}}{(r^2+\varepsilon^2)^{3/2}}\right) = \frac{\vec{\nabla}\cdot\vec{x}}{(r^2+\varepsilon^2)^{3/2}} + \vec{x}\cdot\vec{\nabla}\left(\frac{1}{(r^2+\varepsilon^2)^{3/2}}\right)$$

$$= \frac{3}{(r^2+\varepsilon^2)^{3/2}} - 3\vec{x}\cdot\left(\frac{\vec{x}}{(r^2+\varepsilon^2)^{5/2}}\right)$$

$$= 3\left(\frac{1}{(r^2+\varepsilon^2)^{3/2}} - \frac{r^2}{(r^2+\varepsilon^2)^{5/2}}\right)$$

$$= 3\left(\frac{r^2+\varepsilon^2 - r^2}{(r^2+\varepsilon^2)^{5/2}}\right) = \frac{3\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}}$$

4



$\varepsilon \to 0$

$r \neq 0 \quad \to 0$ as ε^2

$r = 0 \quad \to \infty$ as $\frac{1}{\varepsilon^3}$

How big is this singularity?

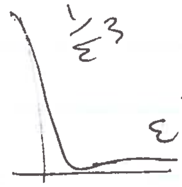
$$\int d^3r \, \frac{3\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}} = \int_0^\infty 4\pi r^2 dr \cdot \frac{3\varepsilon^2}{(r^2+\varepsilon^2)^{5/2}}$$

$$= 4\pi \cdot 3 \cdot \int_0^{\frac{\pi}{2}} \frac{(\tan\theta)^2 \cdot (\sec^2\theta \, d\theta)}{(\tan^2\theta+1)^{5/2}} \cancel{\frac{\varepsilon^2 \cdot \varepsilon^2 \cdot \varepsilon}{\varepsilon^5}}$$

$$= 4\pi \cdot 3 \cdot \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\cancel{\cos^2\theta}} \cdot \frac{1}{\cancel{\cos^4\theta}} \cancel{\cos^5\theta} \, d\theta$$

$$= 4\pi \cdot 3 \cdot \cancel{\int_0^1 (1-\mu^2)} \int_0^1 \underbrace{s^2 ds}_{1/3} = 4\pi$$

$\vec{\nabla}\cdot\left(\frac{\vec{x}}{r^3}\right) \longrightarrow$



$\to \infty \quad r=0.$

$\to 0 \quad r\neq 0$

$\int d^3r = 4\pi$.

$$\boxed{\vec{\nabla}\cdot\left(\frac{\vec{x}}{r^3}\right) = 4\pi\,\delta_D^{(3)}(\vec{x})}$$

Dirac. δ-function 1930.

Heaviside.

Cauchy. 1827 (1D)

Poisson

Result of regularization. $\cancel{\varepsilon^q/(r^2+\varepsilon^2)^{p/2}}$

$$\varepsilon^p/(r^2+\varepsilon^2)^{q/2} \longrightarrow \varepsilon^{3+p-q}$$

$3+p-q>0 \;.\; \to (\varepsilon^{\text{positive}}) \to 0.$ Not really there

$3+p-q<0 \to (\varepsilon^{\text{negative}}) \to \infty$ Bad

$3+p-q = 3+2-5 = 0.$ δ-function

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \rho(\vec{x}') \, \vec{\nabla}\cdot\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \rho(\vec{x}') \left(4\pi \, \delta^{(3)}(\vec{x}-\vec{x}')\right)$$

$$= \frac{\rho(\vec{x})}{4\pi\varepsilon_0} \underbrace{\int d^3x' \, \delta_D^{(3)}(\vec{x}-\vec{x}')}_{1}.$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0}$$

$$\vec{\nabla}\times\vec{E} = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \rho(\vec{x}') \, \vec{\nabla}\times\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)$$

$$\vec{\nabla}\times\left(\frac{\vec{x}}{r^3}\right) = \varepsilon_{ijk}\nabla_j\left(\frac{x^k}{r^3}\right) \qquad \begin{bmatrix} \hat{i} & \hat{j} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

$$(\vec{A}\times\vec{B})_i = \varepsilon_{ijk}A_jB_k \; \cancel{\Sigma \, \varepsilon_{ijk}} \qquad \vec{A}\times\vec{B} = \varepsilon_{ijk}\hat{x}_i A_j B_k$$

ε_{ijk} = odd (antisymmetric)

$\varepsilon_{123} = +1$ 123, 231, 312 $\to +1$

132, 213, 321 $\to -1$

else $\to 0$.

$$\varepsilon_{ijk} = (\hat{x}_i\times\hat{x}_j)\cdot\hat{x}_k = \hat{x}_i\cdot(\hat{x}_j\times\hat{x}_k)$$

$$\vec{\nabla} \times \frac{\vec{x}}{r^3} = \varepsilon_{ijk} (\partial_j) \left(\frac{x_k}{r^3} \right)$$

odd/even

$$= \varepsilon_{ijk} \left(\frac{\delta_{jk}}{r^3} - 3 \frac{x_j x_k}{r^5} \right) = 0$$

$$E_{ij} = E_{ji} \qquad O_{ij} = -O_{ji}$$

$$E_{ij} O_{ij} = E_{pq} O_{pq} = E_{ji} O_{ji} \quad \text{(relabel)}$$

$$= (+E_{ji})(-O_{ji}) \quad \text{(swap)}$$

$$E_{ij} O_{ij} = -E_{ij} O_{ij} \rightarrow 2 E_{ij} O_{ij} = 0 \rightarrow E_{ij} O_{ij} = 0$$

Exactly true even with $\frac{\vec{x}}{(r^2+\varepsilon^2)^{3/2}}$.

$$\vec{\nabla} \times \vec{E} = 0$$