8/28/2015

δ-functions (1D). (Lighthill).

"Distribution". limit of smeared $\delta_\varepsilon(x) = \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2\varepsilon^2}}{\varepsilon}$

Defining property: $\boxed{\int dx\, f(x)\, \delta(x) = f(0)}$

(f well-behaved $x \to \infty$; integrable) $(a,b) \ni 0$

more: change integration variable.

→ Translation: $x' = x - a$ [JDJ 3]. mathworld (2), also (24)

$$\int dx\, f(x)\, \delta(x-a) = \int dx'\, f(x'+a)\, \delta(x') = f(a)$$

→ rescaling $x' = ax$ mathworld (5).

$$\int dx\, f(x)\, \delta(ax) = \int \frac{dx'}{|a|} f\left(\frac{x'}{a}\right) \delta(x') = \frac{1}{|a|} f(0).$$

$a = -1$ $\quad \delta(-x) = \frac{1}{|-1|}\delta(x) = \delta(x)$ $\quad$ (δ is Even)

→ Derivative. $\delta'(x) = \frac{d\delta}{dx}$

$$\int dx\, \delta'(x)\, f(x) = \left[f(x)\,\delta(x)\right]_-^+ - \int dx\, \delta(x)\, f'(x) = -f'(x).$$

JDJ 4 mathworld (19) (10) (18) $\boxed{\delta'(-x) = -\delta'(x)}$ (21) $(\delta'(x)) = 0$

If $g(x)=0$ at $x=x_0$,

$$g(x) \simeq \cancel{g(x_0)} + g'_0 \cdot (x-x_0) + \tfrac{1}{2} g''_0 \cdot (x-x_0)^2 + \cdots$$

$$\int dx\, f(x)\, \delta(g(x)) = \int \frac{dg}{|g'|} f(x)\, \delta(x-x_0) = \frac{1}{|g'_0|} f(x_0).$$

multiple zeroes: $\delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \delta(x-x_i)$

JDJ. 5. mathworld (7).

Ex. $g = x^2 - a^2$ $\quad x_1 = -a \quad x_2 = +a \quad (a>0)$

$$g' = 2x \quad |g'_1| = |g'_2| = 2a$$

$$\delta(x^2-a^2) = \frac{1}{2a}\left[\delta(x-a) + \delta(x+a)\right]$$

$$g = (x-a)(x+a) \qquad \delta((x-a)(x+a)) \simeq \frac{\delta(x+a)}{|x-a|} + \frac{\delta(x-a)}{|x+a|}$$

"best" (perhaps): Gaussian: $\delta_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon^2}} e^{-\frac{1}{2}x^2/\varepsilon^2}$

Height $\sim \frac{1}{\varepsilon}$ width $\sim \varepsilon$ area $= 1$.

$I = \int dx\, e^{-\xi x^2}$? $I^2 = \int dx\, e^{-\xi x^2} \int dy\, e^{-\xi y^2}$

$= \int dx\, dy\, e^{-\xi(x^2+y^2)} = \int \rho\, d\rho \frac{d\phi}{2\pi} e^{-\xi\rho^2} = 2\pi$.

3D: $\delta^{(3)}(\vec{x}-\vec{x}_0) = \lim_{\varepsilon\to 0} \delta_\varepsilon(x-x_0)\, \delta_\varepsilon(y-y_0)\, \delta_\varepsilon(z-z_0)$

$= \frac{1}{(\sqrt{2\pi\varepsilon^2})^3} e^{-\frac{1}{2}\frac{(x-x_0)^2}{\varepsilon^2}} e^{-\frac{(y-y_0)^2}{2\varepsilon^2}} e^{-\frac{1}{2}\frac{(z-z_0)^2}{\varepsilon^2}}$

$= (\quad)^3 e^{-\frac{1}{2}\frac{1}{\varepsilon^2}\left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)}$

$|\Delta\vec{x}|^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta r^2 + r^2\Delta\theta^2 + r^2\sin^2\theta\, \Delta\phi^2$

$\longrightarrow (\quad)^3 \cdot e^{-\frac{1}{2\varepsilon^2}(r-r_0)^2} e^{-\frac{1}{2\varepsilon^2} r^2(\theta-\theta_0)^2} e^{-\frac{1}{2\varepsilon^2} r^2\sin^2\theta(\phi-\phi_0)^2}$

$\delta^{(3)}(\vec{x}-\vec{x}_0) = \delta(r-r_0)\, \delta(r(\theta-\theta_0))\, \delta(r\sin\theta(\phi-\phi_0))$

$\boxed{\delta^{(3)}(\vec{x}-\vec{x}_0) = \frac{1}{r^2\sin\theta} \delta(r-r_0)\, \delta(\theta-\theta_0)\, \delta(\phi-\phi_0)}$

$\int \underline{r^2}dr\, \underline{\sin\theta}\, d\theta\, d\phi$

(4)

cylindrical. $\delta(\rho-\rho_0)\ \delta(\rho(\phi-\phi_0))\ \delta(z-z_0)$

$$= \frac{1}{\rho}\ \delta(\rho-\rho_0)\ \delta(\phi-\phi_0)\ \delta(z-z_0)$$

mathworld (50). $\rho \to 0$? ϕ doesn't matter.

$$\frac{1}{2\pi}\int d\phi'\ \frac{1}{\rho}\ \delta(\rho)\ \delta(\phi-\phi') = \left(\frac{1}{2\pi\rho}\ \delta(\rho)\right) \leftarrow ??$$

mathworld (51) $r \to 0$ θ, ϕ don't matter.

$$\frac{1}{4\pi}\int \sin\theta' d\theta' d\phi'\ \frac{1}{r^2\sin\theta}\ \delta(r)\ \delta(\theta-\theta')\ \delta(\phi-\phi')$$

$$= \frac{1}{4\pi r^2}\delta(r) \leftarrow ??$$

Doesn't happen often (set of measure zero).

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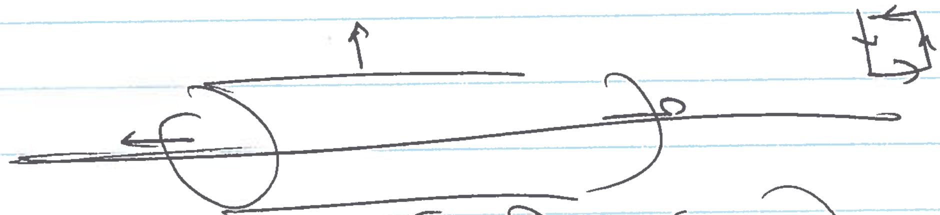
$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0} \quad (1.13) \xleftrightarrow{\text{Divergence theorem}} \oint_S d^2a\, \hat{n}\cdot\vec{E} = Q/\varepsilon_0 \ (\text{in } V)$$

$$\boxed{\vec{\nabla}\times\vec{E} = 0} \quad (1.14) \xleftrightarrow{\text{Stokes theorem}} \oint_{C} \vec{E}\cdot d\vec{l} = 0.$$



wire, revisited

symmetry $\rightarrow$ $\frac{\partial}{\partial z} = 0$ $\quad \frac{\partial}{\partial \phi} = 0$

tube

$$\oint d^2a\, \hat{n}\cdot\vec{E} = (\text{left}) + (\text{tube}) + (\text{right})$$

$$= (-)\int d^2a\, \cancel{E_z(\rho)} + (2\pi\rho)(L)E_\rho + (+)\int d^2a\, \cancel{E_z(\rho)}$$

$$= Q/\varepsilon_0 = \lambda L/\varepsilon_0 \qquad \boxed{E_\rho = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{\rho}}$$

circle

$$\oint \vec{E}\cdot d\vec{l} = (2\pi\rho)\cdot E_\phi = 0 \qquad \boxed{E_\phi = 0}$$

square

$$\int E_z(\rho_1)\, dz + \int \cancel{E_\rho\, d\rho} - \int E_z(\rho_2)\, dz - \int \cancel{E_\rho\, d\rho} = 0$$

$E_z(\rho_1) = E_z(\rho_2)$ $\qquad \boxed{E_z = \text{constant}}$

$\Rightarrow$ Has nothing to do with wire

$\rightarrow$ reflection $E_z = -E_z$

$$\boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{\rho}}{\rho}}$$