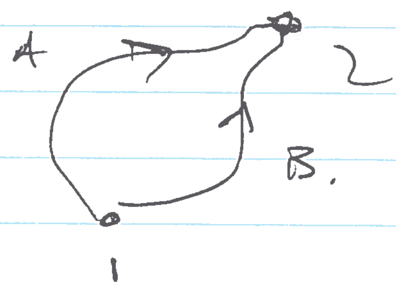
8/31/2015 $\vec{\nabla}\times\vec{E} = 0$: integrability condition: $\vec{E} = -\vec{\nabla}\Phi$.

$$d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy + \frac{\partial\Phi}{\partial z}dz = \vec{\nabla}\Phi\cdot\vec{dl} = -\vec{E}\cdot\vec{dl}$$

$$\Delta\Phi_{12} = -\int_1^2 \vec{E}\cdot\vec{dl}$$



integrable $\rightarrow$ independent of path A/B.

$$\Rightarrow \quad -\int_A \vec{E}\cdot\vec{dl} = -\int_B \vec{E}\cdot\vec{dl}$$

$$\rightarrow \quad \int_{A-B} \vec{E}\cdot\vec{dl} = \oint_C \vec{E}\cdot\vec{dl} = 0$$

$$= \int d^2a\,\hat{n}\cdot(\vec{\nabla}\times\vec{E}) \quad \text{(Stokes theorem)}$$

$\underline{\vec{\nabla}\times\vec{E} = 0}$ + $\underline{\text{Stokes}}$ $\rightarrow$ $\underline{\oint = 0}$ $\rightarrow$ indep of path

$\rightarrow$ $\underline{\text{integrable}}$, $\rightarrow \Phi$

$$\vec{E} = -\vec{\nabla}\Phi = -\nabla_i\Phi$$

$$\rightarrow (\vec{\nabla}\times\vec{E})_i = \varepsilon_{ijk}\nabla_j E_k = -\varepsilon_{ijk}\underline{\nabla_j\nabla_k}\Phi$$

$\checkmark$ Automatic. (second exterior derivative $d^2=0$)

integral gives $\Delta\Phi$ from some reference point.

$$\Phi(\vec{x}) = -\int_{\vec{x}_0}^{\vec{x}} \vec{E}\cdot\vec{dl} \qquad \Phi(\vec{x}_0) = 0.$$

different point: $\Phi_0 = -\int_{x_0}^{\vec{x}} \vec{E}\cdot\vec{dl}$

$$\Phi_1 = -\int_{x_1}^{\vec{x}} \vec{E}\cdot\vec{dl}$$

$$\Phi_1 - \Phi_0 = -\int_{x_1}^{x} + \int_{x}^{x_0} = -\int_{x_1}^{x_0} \vec{E}\cdot\vec{dl} = \text{constant}$$

doesn't matter. same $\vec{E}$. (gauge).

note. $\vec{\nabla}\frac{1}{r} = -\frac{\vec{x}}{r^3} \quad (p=-1)$

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \rho(\vec{x}') \left[-\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}\right]$$

$$= -\vec{\nabla}\left(\frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)$$

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}} \quad (1.17)$$

$\vec{E} = -\vec{\nabla}\Phi \qquad \cancel{\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 = -\vec{\nabla}}$

$$\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot(-\vec{\nabla}\Phi) = -\nabla^2\Phi = \rho/\varepsilon_0. \qquad \boxed{\nabla^2\Phi = -\rho/\varepsilon_0}$$

(1.28)

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}.$$

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ (Back cover)

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Example: Long straight wire.

$$\Phi(\rho,\phi,z) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\,dz'}{(\rho^2 + \cancel{\rho'^2} - \cancel{2\rho\rho'\cos(\phi-\phi')} + (z-z')^2)^{1/2}}$$

$$\Phi(\rho) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\,dz''}{(\rho^2+z''^2)^{1/2}} \qquad (-\infty,\infty) \to -L, L.$$

$$\boxed{\Phi = \frac{\lambda}{4\pi\varepsilon_0}\log\left[\frac{\sqrt{L^2+\rho^2}+L}{\sqrt{L^2+\rho^2}-L}\right]}$$

$$z = \rho\tan\theta \to 2\log(\sec\theta + \tan\theta) \qquad \frac{e^x - e^{-x}}{2} = \sigma.$$

$$z = \rho\sinh\theta \to 2\sinh^{-1}\left(\frac{\rho}{L}\right) 2\log\left(\sqrt{1+\frac{L^2}{\rho^2}} + \frac{L}{\rho}\right)$$

check numerics
$\log(-L+\sqrt{L^2})$

$$\to \log\left[\frac{(\sqrt{L^2+\rho^2}+L)^2}{(L^2+\rho^2) - \cancel{L^2}}\right] = 2\log\left(\sqrt{L^2+\rho^2}+L\right) - 2\log\rho$$

$$= 2\log L + 2\log\left(\sqrt{1+\frac{\rho^2}{L^2}}+1\right) - 2\log\rho$$

$$\simeq 2\log L - 2\log\rho + 2\log 2 + 2\left(\frac{r^2}{4L^2}\right)$$

(4)

$$\Phi \simeq \frac{\lambda}{2\pi\varepsilon_0}\left(\underbrace{\log L + \log 2}_{\text{constant}} - \log\rho + \frac{1}{4}\frac{\rho^2}{L^2} + \cdots\right)$$

$\frac{1}{4}\frac{\rho^2}{L^2} \to 0$

$$\boxed{\Phi \simeq -\frac{\lambda}{2\pi\varepsilon_0}\ln\rho} + \text{constant}.$$

$$\vec{E} = -\vec{\nabla}\Phi = -\hat{\rho}\frac{\partial\Phi}{\partial\rho} = +\frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{\rho}}{\rho}.$$

renormalization counterterms.

Dimensional regularization

$$\lambda = \lambda_0\left|\frac{z}{a}\right|^{-\varepsilon} \qquad \varepsilon \to 0^+$$

$$\Phi(z=0) = \frac{1}{4\pi\varepsilon_0}\int\frac{dz'\,\lambda_0\left|\frac{z'}{a}\right|^{-\varepsilon}}{\sqrt{\rho^2+z'^2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\lambda_0\left(\frac{\rho}{a}\right)^{-\varepsilon}\frac{\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma\left(1-\frac{\varepsilon}{2}\right)}{\sqrt{\pi}}.$$

$$\boxed{\Gamma(x) = \int_0^\infty t^{x-1}\,dt\,e^{-t}} \qquad \Gamma(x) = (x-1)\Gamma(x-1) \text{ (parts)}$$

$$\Gamma(n) = (n-1)!$$

$$\boxed{\Gamma(1) = 1.}$$

$$\lim_{N\to\infty}\left(\sum_{i=1}^{N}\frac{1}{i} - \log N\right)$$

$$\downarrow\ 0.5772156649$$

$$\Gamma\left(1+\frac{\varepsilon}{2}\right) \overset{?}{=} \frac{\varepsilon}{2}\,\Gamma\left(\frac{\varepsilon}{2}\right)$$

$$\rightarrow \Gamma\left(\frac{\varepsilon}{2}\right) = \frac{\Gamma\left(1+\frac{\varepsilon}{2}\right)}{(\varepsilon/2)} = \frac{1 - \gamma_E\cdot\frac{\varepsilon}{2} + O(\varepsilon^2)}{\varepsilon/2}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

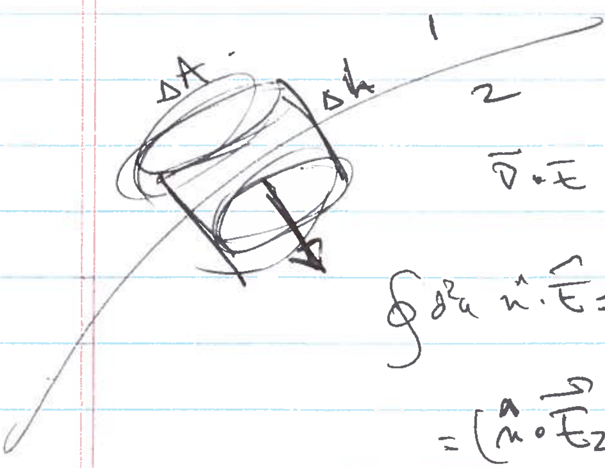
$$\Gamma\left(\frac{1-\varepsilon}{2}\right) = \sqrt{\pi}\left(1 + \frac{\varepsilon}{2}\left(\gamma_E + \log 4\right) + O(\varepsilon^2)\right)$$

$$\left(\frac{\rho}{a}\right)^{-\varepsilon} = e^{-\varepsilon\log(\rho/a)} = 1 - \varepsilon\log\left(\frac{\rho}{a}\right) + O(\varepsilon^2)$$

$$\Phi = \frac{\lambda_0}{2\pi\varepsilon_0}\left(\frac{2}{\varepsilon} - 2\log\left(\frac{\rho}{2a}\right) + O(\varepsilon)\right)$$

$$\Phi(\rho) - \Phi(b) = \lim_{L\to\infty}\frac{\lambda}{4\pi\varepsilon_0}\log\left[\frac{\left(\sqrt{L^2+\rho^2}+L\right)^2}{\left(\sqrt{L^2+b^2}+L\right)^2}\cdot\frac{b^2}{\rho^2}\right]$$

Boundary conditions.



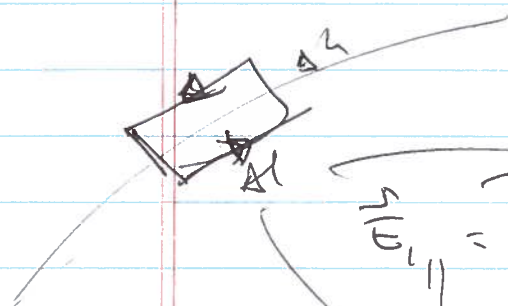
$$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 .$$

$$\oint d^2a\ \hat{n}\cdot\vec{E} = (\text{top}) + (\text{sides}) + \text{bottom}.$$

$$= (\hat{n}\cdot\vec{E}_2)(\Delta A) + \cancel{(\text{sides})\Delta h} + (-\hat{n})\cdot\vec{E}_1\,\Delta A$$

$$= Q/\varepsilon_0 = \sigma\cdot\Delta A/\varepsilon_0 .$$

$$\boxed{\hat{n}\cdot(\vec{E}_2-\vec{E}_1) = \sigma/\varepsilon_0}$$



$$\oint \vec{E}\cdot\vec{dl} = (E_1 - E_2)\,\Delta l + \cancel{(\ )\Delta h} = 0$$

$$\vec{E}_{1_\parallel} = \vec{E}_{2_\parallel} \qquad \boxed{\hat{n}\times(\vec{E}_2-\vec{E}_1) = 0}$$

$$\Delta\Phi = \int_1^2 E\,dx = E_\perp\cdot\Delta h \quad \to 0 .$$

Dipole layer: parallel plate. $E = \sigma/\varepsilon_0$.

$\sigma\to\infty$ $D = \sigma\Delta h = \text{constant}.$ $\Delta\Phi = D/\varepsilon_0$

$\Delta h \to 0$

biological lipid bilayer.