9|4|2015

Theorems Green's Functions

(UFOs, depopulation of planet earth.)

George Green

$$\int_V d^3x \; \vec{\nabla}\cdot\vec{V} = \oint_S d^2a \; \hat{n}\cdot\vec{V} \qquad \left(\int_\Omega d\omega = \int_{\partial\Omega} \omega \right).$$

let $\vec{V} = \phi \nabla \psi$ $\qquad \vec{\nabla}\cdot\vec{V} = \phi \nabla^2 \psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi$

$\hat{n}\cdot\vec{V} = \phi \; \hat{n}\cdot\vec{\nabla}\psi = \phi \frac{\partial \psi}{\partial n}.$

$$\boxed{\int_V d^3x \left[\phi \nabla^2 \psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right] = \oint_S d^2a \; \phi \frac{\partial \psi}{\partial n}}$$

Green's 1st. Identity.

$\phi \leftrightarrow \psi$, subtract.

$$\boxed{\int_V d^3x \left[\phi \nabla^2 \psi - \psi \nabla^2 \phi\right] = \oint_S d^2a \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right)}$$

Green's 2nd Identity, Green's Theorem (1824).

Essay (1828). arXiv

google. (p.12)

Let : $\underline{\phi = \Phi}$ $\quad \psi = \frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{R}|} = \frac{1}{R}$ $\quad \underrightarrow{\vec{x} \to \vec{x}'}$

$$\nabla^2 \psi = \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = \nabla'^2 \frac{1}{|\vec{x}-\vec{x}'|} = +\nabla' \cdot \left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|} \right) = -4\pi \delta^{(3)}(\vec{x}-\vec{x}').$$

$$\int_V d^3x' \left[\Phi(\vec{x}') \left(-4\pi \delta^{(3)}(\vec{x}-\vec{x}') \right) - \frac{1}{|\vec{x}-\vec{x}'|} \left(-\rho/\varepsilon_0 \right) \right]$$

$$= -4\pi \Phi(\vec{x}) + \frac{1}{\varepsilon_0} \int_V d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} = \oint_S d^2a' \left(\Phi \frac{\partial}{\partial n'} \frac{1}{R} - \frac{1}{R} \frac{\partial \Phi}{\partial n'} \right)$$

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|} + \frac{1}{4\pi} \oint_S d^2a' \left(\frac{1}{R} \frac{\partial \Phi}{\partial n'}\Big|_S - \Phi\Big|_S \frac{\partial}{\partial n'} \frac{1}{R} \right)}$$

(1.36)

$\Phi(\vec{x})$ needs ρ in V $\quad$ $\Phi|_S, \frac{\partial \Phi}{\partial n}\Big|_S$ on S $\quad (E_n)$

in fact : $\Phi|_S$ or $\frac{\partial \Phi}{\partial n}\Big|_S$, is sufficient.

unique

Φ_1, Φ_2 $\quad \nabla^2 \Phi = -\rho/\varepsilon_0$, $\quad \Phi|_S, \frac{\partial \Phi}{\partial n}\Big|_S$. $\quad$ Same let $U = \Phi_1 - \Phi_2$.

Apply 1st identity, $\phi = \psi = U$.

$$\int_V d^3x \left(U \cancel{\nabla^2 U} + \vec{\nabla} U \cdot \vec{\nabla} U \right) = \oint_S d^2a \; U \cdot \frac{\partial U}{\partial n}.$$

Boundary condition. $\Phi|_S$ given $\Rightarrow U|_S = 0 \rightarrow$ RHS $= 0$.

or. $\quad \frac{\partial \Phi}{\partial n}\Big|_S \rightarrow \frac{\partial U}{\partial n}\Big|_S = 0 \rightarrow$ RHS $\rightarrow$

either way: $\int_V d^3x \, |\vec{\nabla} U|^2 = 0$.

$\Rightarrow \vec{\nabla} U = 0$ in V $\qquad U|_S = 0 \rightarrow U = 0$.

$\frac{\partial U}{\partial n}\Big|_S = 0 \rightarrow U = $ constant

$\Phi_1 - \Phi_2 =$ constant

Green's Function. 2 kinds. Dirichlet

Neumann.

$\nabla^2 G = -4\pi \, \delta^{(3)}(\vec{x} - \vec{x}') \qquad G_D\Big|_S = 0$.

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V d^3x' \, \rho(\vec{x}') \, G_D(\vec{x}, \vec{x}') - \frac{1}{4\pi} \oint d^2a' \, \Phi|_S \frac{\partial G}{\partial n'}} \quad (1.44)$$

Neumann: want to impose $\left.\frac{\partial G}{\partial n'}\right|_{\vec{x}' \text{ on } S} = 0$

Can't $\int_V d^3x\, \vec{\nabla}\cdot\vec{V} = \oint_S da'\, \hat{n}' \cdot \vec{V}$ $\qquad \vec{V} = \vec{\nabla} G$

$$\int d^3x'\, \nabla'^2 G = \int d^3x' \left(-4\pi\delta(\vec{x}-\vec{x}')\right) = -4\pi = \oint da' \frac{\partial G}{\partial n'}.$$

Next best: take $\left.\frac{\partial G}{\partial n'}\right|_S =$ constant $\rightarrow (S)\left(\frac{\partial G}{\partial n'}\right) = -4\pi$

$$\left.\left(\frac{\partial G}{\partial n'}\right)\right|_S = \frac{-4\pi}{S}$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x' \frac{\rho(\vec{x}')}{\cancel{|\vec{x}-\vec{x}'|}} G_N(\vec{x},\vec{x}') + \frac{1}{4\pi}\oint_S d^2a' \left[G_N \frac{\partial \Phi}{\partial n'} + \left(+\frac{4\pi}{S}\right)\Phi|_S \right]$$

$$\boxed{\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x'\, \rho(\vec{x}')\, G_N + \frac{1}{4\pi}\oint_S d^2a'\, G_N \frac{\partial \Phi}{\partial n'} + \langle \Phi \rangle_S}$$

(1.46)

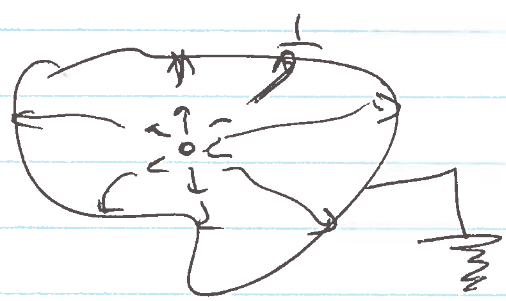
A) $G_D : (\vec{x}, \vec{x}')$

mathematical object.

$$\nabla^2 G = -4\pi\, \delta(\vec{x}-\vec{x}') \qquad G(x')\Big|_S = 0.$$

B) Physics G is potential of point charge.
"q" $= 4\pi\epsilon_0$

in presence of grounded, conducting S.



C) Inverse of laplacian.

$$\nabla^2\Phi = -\rho/\epsilon_0. \qquad \Phi = (\nabla^2)^{-1}(\rho/\epsilon_0) + \Phi_0$$

$\nabla^2\Phi_0 = 0$ in V.

Symmetries. images.
Systematics: orthogonal functions.