9/9/2015

Dirichlet

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \, \rho(\vec{x}')\, G_D(\vec{x},\vec{x}') - \frac{1}{4\pi}\oint_S da' \, \Phi\big|_S \frac{\partial G_D}{\partial n'}$$

(1.44)

Neumann

$$\Phi = \ldots + \frac{1}{4\pi}\oint_S da' \, \frac{\partial \Phi}{\partial n'} \cdot G_N + \langle \Phi \rangle_S$$

(1.46)

Defining G

$$\nabla'^2 G(\vec{x},\vec{x}') = -4\pi\, \delta^{(3)}(\vec{x}-\vec{x}')$$

$$G_D\big|_S = 0 \qquad \frac{\partial G_N}{\partial n'}\bigg|_S = -\frac{4\pi}{S}$$

Green's Theorem

$$\int_V d^3x' \left[\phi \nabla'^2 \psi - \psi \nabla'^2 \phi\right] = \oint da' \left(\phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'}\right)$$

Let $\phi = G_D(\vec{x},\vec{x}')$ $\quad \psi = G_D(\vec{y},\vec{x}')$

$$\int_V d^3x' \left[G(\vec{x},\vec{x}')\left(-4\pi\,\delta(\vec{y}-\vec{x}')\right) - G(\vec{y},\vec{x}')\left(-4\pi\,\delta(\vec{x}-\vec{x}')\right)\right]$$

$$= \oint da' \left(G_D(\vec{x},\vec{x}') \frac{\partial G_D}{\partial n'}(\vec{y},\vec{x}') - \ldots \right)$$

$$-4\pi\left[G(\vec{x},\vec{y}) - G(\vec{y},\vec{x})\right] = 0$$

$$G_D(\vec{x},\vec{y}) = G_D(\vec{y},\vec{x})$$

Neumann — JDJ

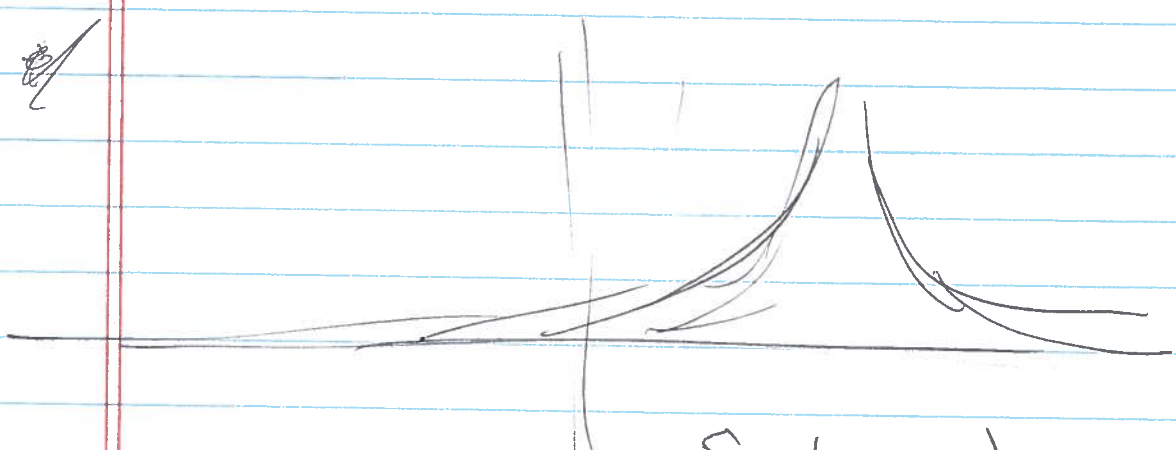
Chapter 2 Examples -

$V = \{z \geq 0\}$ $S' = \{z = 0\}$

$\nabla^2 G = -4\pi\, \delta(\vec{x} - \vec{x}') \rightarrow \dfrac{1}{|\vec{x} - \vec{x}'|}$

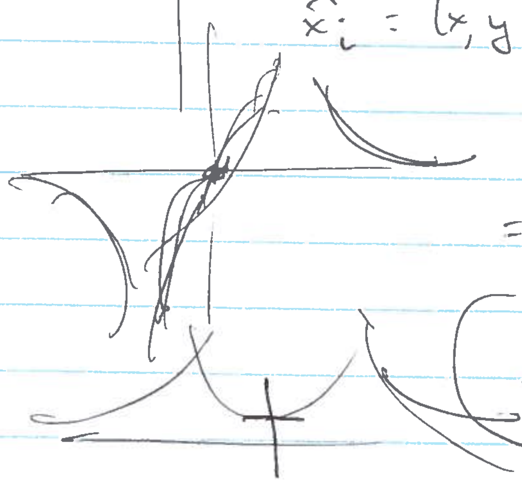
only interior of V matters.

$\rightarrow G = \dfrac{1}{|\vec{x}-\vec{x}'|} +$ [charges that may or may not exist outside V]



$\vec{x} = (x, y, z)$

$\hat{x}_i = (x, y, -z)$



$q_i = -q$

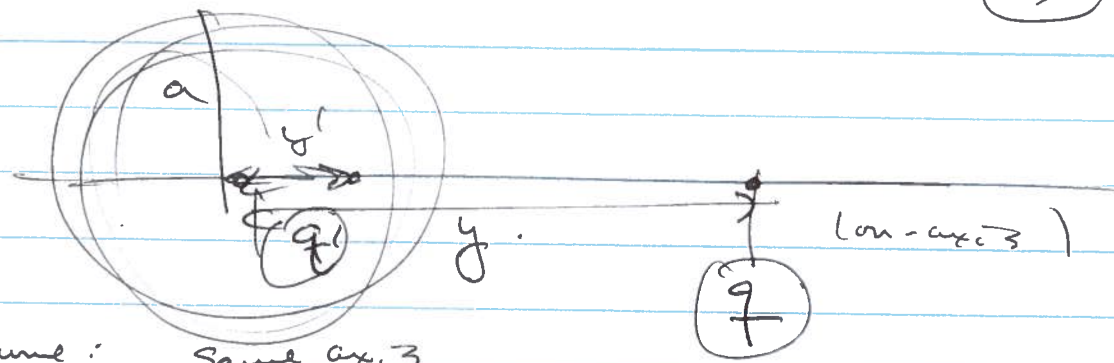
$@ z' = 0, \; d = d_i$

$= \sqrt{(x-x')^2 + (y-y')^2 + z^2}$

$\dfrac{\partial G}{\partial z'} = 0 \quad @ z' = 0$

$q_i = +q$

$\cdot\ \vec{x} = (r, \theta, \phi)$.



Assume: Same axis

$$\Phi(r,\theta,\phi) = \frac{1}{4\pi\varepsilon_0}\frac{q}{(r^2+y^2-2ry\cos\theta)^{1/2}} + \frac{1}{4\pi\varepsilon_0}\frac{q'}{(r^2+y'^2-2ry'\cos\theta)^{1/2}}$$

seek q', y' such that $\Phi = 0$ @ $r = a$.

$$\frac{q}{(a^2+y^2-2ay\cos\theta)^{1/2}} + \frac{q'}{(a^2+y'^2-2ay'\cos\theta)} = 0$$

$$q^2(a^2+y'^2-2ay'\cos\theta) = q'^2(a^2+y^2-2ay\cos\theta)$$

constant $\quad q^2(a^2+y'^2) = q'^2(a^2+y^2)$

$\cos\theta \quad q^2 y' = q'^2 y \rightarrow q'^2 = q^2\, y'/y$

opposite in sign

$$q^2(a^2+y'^2) = q^2\frac{y'}{y}(a^2+y^2)$$

$$y(a^2+y'^2) = y'(a^2+y^2)$$

quadratic

(4)

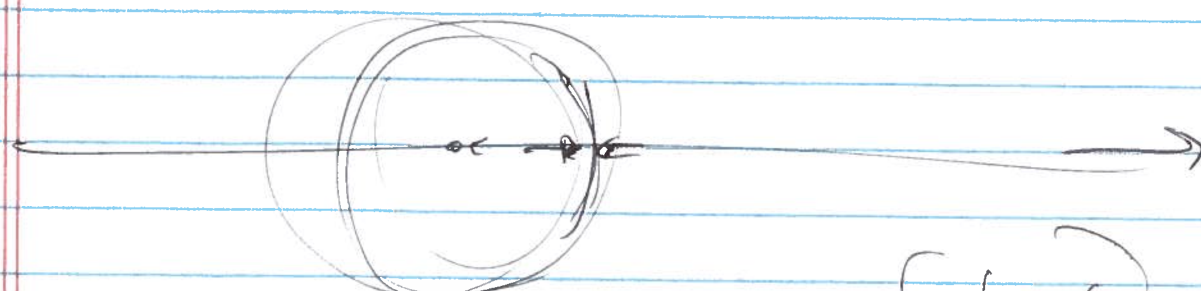
$$y y'^2 - (a^2+y^2) y' + y a^2 = 0$$

$$y' = \frac{(a^2+y^2) \pm \sqrt{(a^2+y^2)^2 - 4 \cdot y \cdot y a^2}}{2y}$$

$$= \frac{(a^2+y^2) \pm (a^2-y^2)}{2y}$$

(−). $y' = y$ $\quad q' = -q$ $\quad \varphi = 0$.

(+). $y' = a^2/y$. $\quad q' = -q\, a/y$



$y = a + \varepsilon$ $\quad y' = \frac{a^2}{a+\varepsilon} = \frac{a}{(1+\varepsilon/a)} = a(1 - \varepsilon/a) = a - \varepsilon$ $\quad q' \to -q$

$y \to 0.$ $\quad y' \to 0.$ $\quad q' \to 0,$

$$G_D = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{(a^2/r')}{\left|\vec{x} - \frac{a^2}{r'}\hat{r}'\right|}$$

$r'\hat{r}'$

$y \to r'$

z axis → arbitrary direction

[potential of ($q = 4\pi\epsilon_0$) @ (r') + ($q' = -\frac{a}{r'}q$) @ $\frac{a^2}{r'}$]

$$|\vec{x}-\vec{x}'|^2 = \vec{x}\cdot\vec{x} + \vec{x}'\cdot\vec{x}' - 2\vec{x}\cdot\vec{x}'$$

$$= r^2 + r'^2 - 2rr'\cos\gamma$$

$$G_D = \frac{1}{(r^2+r'^2-2rr'\cos\gamma)^{1/2}} - \frac{a/r'}{\left(r^2 + \left(\frac{a^2}{r'}\right)^2 - 2r\frac{a^2}{r'}\cos\gamma\right)^{1/2}}$$

(2.17)

$$G_D = \frac{1}{(r^2+r'^2-2rr'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{r^2r'^2}{a^2} + a^2 - 2rr'\cos\gamma\right)^{1/2}}$$

$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

$$\hat{r}\cdot\hat{r}' = \sin\theta\cos\phi\sin\theta'\cos\phi' + \sin\theta\sin\phi\cdot\sin\theta'\sin\phi' + \cos\theta\cos\theta'$$

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

V = exterior = $\{r > a\}$.

$\hat{n}' = -\hat{r}'$ (6)

$$-\frac{\partial G}{\partial n'} = \frac{\partial G}{\partial r'} = \left(-\tfrac{1}{2}\right)\left(r^2 + r'^2 - 2rr'\cos\gamma\right)^{-3/2}\left(2r' - 2r\cos\gamma\right)$$

$$+ \left(+\tfrac{1}{2}\right)\left(\frac{r^2 r'^2}{a^2} + a^2 - 2rr'\cos\gamma\right)^{-3/2}\left(\frac{2r^2 r'}{a^2} - 2r\cos\gamma\right)$$

$$-\frac{\partial G}{\partial n'}\Big|_{r'=a} = \frac{-1}{\left(r^2+a^2-2ar\cos\gamma\right)^{3/2}}\left(a - r\cos\gamma\right) + \frac{\left(r^2/a - r\cos\gamma\right)}{\left(r^2+a^2-2ar\cos\gamma\right)^{3/2}}$$

$$= \frac{(r^2-a^2)/a}{\left(r^2+a^2-2ar\cos\gamma\right)^{3/2}}$$

Boundary term

$$\Phi(r,\theta,\varphi) = \frac{1}{4\pi}\oint a^2 d\Omega' \, V_S(\theta',\varphi') \frac{(r^2-a^2)}{a\left(r^2+a^2-2ar\cos\gamma\right)^{3/2}}$$

(2.19)