9/11/2015

V = exterior of sphere = $\{r>a\}$.

$$G = \frac{1}{(r^2+r'^2-2rr'\cos\gamma)^{1/2}} - \frac{1}{\left(\frac{r^2r'^2}{a^2}+a^2-2rr'\cos\gamma\right)^{1/2}}$$

$\left(\cos\gamma = \hat{r}\cdot\hat{r}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right)$

$\cos\gamma=1$, $r=r'$; $r'=r$; $r=a^2/r'$, $r'=a^2/r$; in/out

$\nabla^2 G = \nabla'^2 G = -4\pi\,\delta^{(3)}(\vec{x}-\vec{x}')$ (1st term) $(r'=r,\ \cos\gamma=1)$

$G|_{r=a} = G|_{r'=a} = 0$. Symmetric.

$$\Phi = -\frac{1}{4\pi}\int a^2 d\Omega'\, V_s(\theta',\phi')\,\frac{\partial G}{\partial n'}$$

$$\Phi(r,\theta,\phi) = \int \frac{d\Omega'}{4\pi}\,\frac{a(r^2-a^2)}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\,V_s(\theta',\phi')$$

(2.19).

$V = \{r<a\}$. Same G. $\hat{n}' = +\hat{r}'$

$(r^2-a^2) \to (a^2-r^2)$ (so it's >0)

(2)

$r \to 0$

$$\Phi(r=0) = \int \frac{d^2\Omega'}{4\pi} \frac{a(a^2-r^2)}{(r^2+a^2-2ar\cos\gamma)^{3/2}} V_S(\theta',\phi') = \langle \Phi \rangle_S$$

a^3

$(a^2)^{3/2}$

$r \to a$, $a^2 - r^2 \to 0$ But denominator $\to 0$ @ $\cos\gamma \to 1$.

let $r = a(1+\varepsilon)$ $\quad \theta' = \theta + \Delta\theta \quad \phi' = \phi + \Delta\phi$.

$$\cos\gamma = \cos\theta\cos(\theta+\Delta\theta) + \sin\theta\sin(\theta+\Delta\theta)\cos(\Delta\phi).$$

$$= \cos^2\theta - \Delta\theta\sin\theta\cos\theta - \tfrac{1}{2}\Delta\theta^2\cos^2\theta + \left(\sin^2\theta + \Delta\theta\sin\theta\cos\theta - \tfrac{1}{2}\Delta\theta^2\sin^2\theta\right)\left(1 - \tfrac{1}{2}\Delta\phi^2\right)$$

$$\simeq 1 - \tfrac{1}{2}\left((\Delta\theta)^2 + \sin^2\theta\,(\Delta\phi)^2\right).$$

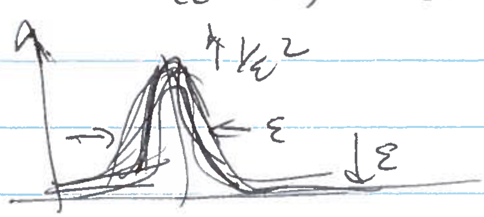
$$\frac{a(a^2-r^2)}{(r^2+a^2-2ar\cos\gamma)^{3/2}} \longrightarrow \frac{\varepsilon(2+\cancel{\varepsilon})}{\left(\varepsilon^2 + 2(1+\cancel{\varepsilon})(1-\cos\gamma)\right)^{3/2}}$$

$$\Phi(a,\theta,\phi) \to \int \frac{d\Omega'}{4\pi} V_S(\theta',\phi') \frac{2\varepsilon(1+\cancel{\varepsilon/2})}{\left(\varepsilon^2 + (1+\cancel{\varepsilon})(\Delta\theta^2 + \sin^2\theta\,\Delta\phi^2)\right)^{3/2}}$$

$$\frac{2\varepsilon}{\left[\varepsilon^2 + (\Delta\theta^2 + \sin^2\theta\,\Delta\phi)^2\right]^{3/2}} \to \begin{cases} 0 & \Delta\theta \neq 0. \quad (\varepsilon) \\ \infty & \Delta\theta = \Delta\phi = 0. \quad \left(\frac{1}{\varepsilon^2}\right) \end{cases}$$

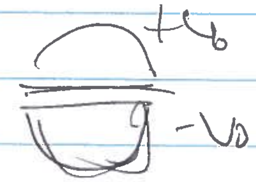
$$\int d(\Delta\theta)\, d(\Delta\phi) \frac{2\varepsilon}{(\varepsilon^2 + \dots)} = \frac{4\pi}{\sin\theta}$$

$\delta(\Omega - \Omega')$



Back to the outside

$$V_s = V_0 \cdot \mathrm{sgn}(\cos\theta') = \begin{cases} +V_0 & (\cos\theta > 0) \\ -V_0 & (\cos\theta < 0) \end{cases}$$



[$\sum$ now ; $P_\ell(\cos\theta)$; radiation (dipole)]

Can't do the integral in general.

Try first on-axis $\quad \cos\theta = \pm 1 \quad \sin\theta = 0 \quad \cos\gamma = \pm\cos\theta'$

Above ($\cos\theta = +1$) $\quad \Phi(r, \theta = 0) = \frac{a(r^2 - a^2)}{4\pi} V_0 \int_{-1}^{1} \frac{d\mu' \cdot 2\pi \cdot \mathrm{sgn}\,\mu'}{(r^2 + a^2 - 2ar\mu')^{3/2}}$

$z^2 = r^2 + a^2 - 2ar\mu' \qquad z\,dz = ar\,d\mu' \; (-)$

$$\Phi = \frac{1}{2} \frac{\cancel{a}(r^2 - a^2) V_0}{\cancel{a} r} \left[\int_0^1 \frac{d\mu'}{(z^2)^{3/2}} + \int_{-1}^{0} \frac{d\mu' \, (-)}{(z^2)^{3/2}} \right]$$

$$[\;] = \int_{r-a}^{\sqrt{r^2+a^2}} \frac{z\,dz}{z^3} - \int_{\sqrt{r^2+a^2}}^{r+a} \frac{z\,dz}{z^3}$$

$$= \left(\frac{1}{z}\right)_{r-a}^{\sqrt{r^2+a^2}} - \left(\frac{1}{z}\right)_{\sqrt{r^2+a^2}}^{r+a}$$

$$= \frac{-1}{\sqrt{r^2+a^2}} + \frac{1}{r-a} + \frac{1}{r+a} - \frac{1}{\sqrt{r^2+a^2}}$$

$$\Phi = V_0 \cdot \frac{a(r^2-a^2)}{r\,a}\left[\frac{r}{r^2-a^2} - \frac{1}{\sqrt{r^2+a^2}}\right]$$

$$\boxed{\Phi = V_0\left[1 - \frac{r^2-a^2}{r\sqrt{r^2+a^2}}\right]}$$

$(\cos\theta) < 0 \longrightarrow -$ odd

$r \to \infty$ $\Phi = V_0$

$r \to 0$ $(+)$ $\Phi \simeq V_0\left(\frac{3}{2}\frac{a^2}{r^2} - \frac{7}{8}\frac{a^4}{r^4} + \frac{11}{16}\frac{a^6}{r^6} - \frac{75}{128}\frac{a^8}{r^8} + \cdots\right)$

(will remind you)

otherwise, approximate : large r (small r) small $\cos\gamma$

$$\frac{1}{(r^2+a^2-2ar\cos\gamma)^{3/2}} = \frac{1}{(r^2+a^2)^{3/2}} \frac{1}{\left(1-\frac{2ar\cos\gamma}{r^2+a^2}\right)^{3/2}}$$

$$\hookrightarrow (1-\alpha\cos\gamma)^{-3/2}$$

$\alpha \le 1$ $\alpha = 1$ only @ $r=a$.

$\frac{315}{128}$

$$\frac{1}{(1-\alpha\cos\gamma)^{3/2}} \simeq 1 + \frac{3}{2}\alpha\cos\gamma + \frac{15}{8}\alpha^2\cos^2\gamma + \frac{35}{16}\alpha^3\cos^3\gamma + \cdots$$

$$\Phi = \frac{V_0 a(r^2-a^2)}{4\pi} \frac{1}{(r^2+a^2)^{3/2}} \int d\Omega' \left(1+\frac{3}{2}\alpha\cos\gamma + \cdots\right) \operatorname{sgn}\mu'.$$

$$\int d\Omega' \operatorname{sgn}\mu' = 2\pi \cdot \int_{-1}^{1} \operatorname{sgn}\mu' \, d\mu' = 0$$

$$\int d\Omega' \left(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\,\cos(\phi-\phi')\right) \operatorname{sgn}\mu' \quad \overset{(2.128)}{=} \boxed{2\pi\cos\theta}$$

$$= 2\pi\cdot\mu \int_{-1}^{1} d\mu' \cdot \mu'\operatorname{sgn}\mu' = 2\pi\mu\cdot 2\int_0^1 \mu\, d\mu = 2\pi\mu$$

$$\int d\Omega' \left(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')\right)^2 \operatorname{sgn}\mu' = 0.$$

$$= \int d\Omega' \left(\mu^2\mu'^2 + 2\mu\mu'\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi') + (1-\mu^2)(1-\mu'^2)\cos^2(\phi-\phi')\right) \operatorname{sgn}\mu'$$