$\frac{315}{128}$

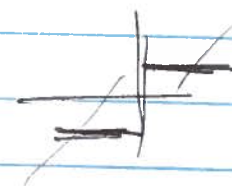
9/14/2015

$$\Phi(r,\theta,\phi) = \frac{V_0 a (r^2 - a^2)}{4\pi (r^2+a^2)^{3/2}} \int d\Omega' \left(1 + \frac{3}{2}\alpha \cos\gamma + \frac{15}{8}\alpha^2 \cos^2\gamma + \frac{35}{16}\alpha^3 \cos^3\gamma\right) \times \operatorname{sgn}\mu'$$

$$\underline{\cos\gamma} = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$
$$= \mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')$$

$$\underline{d\Omega'} = d\mu' d\phi'$$

$$\int d\Omega' \cdot \operatorname{sgn}\mu' = 2\pi \int_{-1}^{1} \operatorname{sgn}\mu' \, d\mu' = \underline{0}$$



$$\int d\Omega' \left(\mu\mu' + \cancel{\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')}^{\,0}\right)\operatorname{sgn}\mu'$$
$$= 2\pi\mu \int_{-1}^{1} d\mu' \, \mu' \operatorname{sgn}\mu' = 2\pi\mu \cdot 2\underbrace{\int_0^1 d\mu' \mu'}_{2\cdot\frac{1}{2}} = \underline{\underline{2\pi\mu}}$$

$$\int d\Omega' \left(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')\right)^2 \operatorname{sgn}\mu'$$
$$= \int d\Omega' \left(\cancel{\mu^2\mu'^2} + \cancel{2\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')} + \cancel{(1-\mu^2)(1-\mu'^2)}\cos^2(\phi-\phi')\right)\operatorname{sgn}\mu' = \underline{0}$$

2

$$\int d\Omega' \left(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi')\right)^3 \operatorname{sgn}\mu'$$

$$= \int d\Omega' \Big[\left(\mu^3\mu'^3 + 3\mu^2\mu'^2\sqrt{\ }\sqrt{\ }\cos(\phi-\phi')\right) + 3\mu\mu'(1-\mu^2)(1-\mu'^2)\cos^2(\phi-\phi') + \cdots \cos^3(\phi-\phi') \Big]$$

$\to 0$

$$= 2\pi\mu^3 \cdot \underbrace{2\int_0^1 d\mu'\,\mu'^3}_{2\cdot\frac{1}{4}} + 3\cdot 2\pi\,\mu(1-\mu^2)\cdot\tfrac{1}{2}\cdot \underbrace{2\int_0^1 d\mu'\,\mu'(1-\mu'^2)}_{2\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{1}{2}}$$

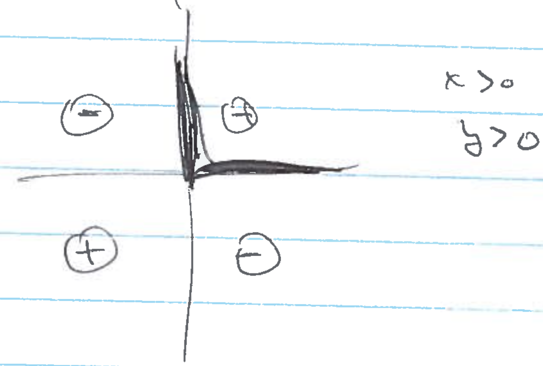
$$= \pi\mu^3 + \frac{3\pi}{2}\mu(1-\mu^2) = \frac{3\pi}{2}\cos\theta - \frac{\pi}{2}\cos^3\theta \qquad (2.25)$$

$$\Phi = \frac{V_0 a(r^2-a^2)}{4\pi(r^2+a^2)^{3/2}} \left[\frac{3}{2}\cdot\frac{2ar}{r^2+a^2}\cdot 2\pi\cos\theta + \frac{35}{16}\cdot\left(\frac{2ar}{r^2+a^2}\right)^3 \left(\frac{3\pi}{2}\cos\theta - \frac{\pi}{2}\cos^3\theta\right) + \cdots \right]$$

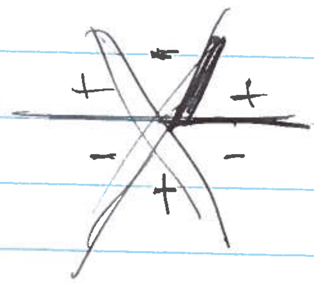
$$\Phi = \frac{V_0 a(r^2-a^2)}{(r^2+a^2)^{3/2}} \left(\frac{3}{2}\frac{ar}{r^2+a^2}\cos\theta + \frac{35}{16}\left(\frac{ar}{r^2+a^2}\right)^3 (3\cos\theta - \cos^3\theta) + \cdots \right) \qquad (2.26)$$

Images → half space. $z>0$, $z<0$.
interior/exterior of sphere $a>r$, $r>a$.

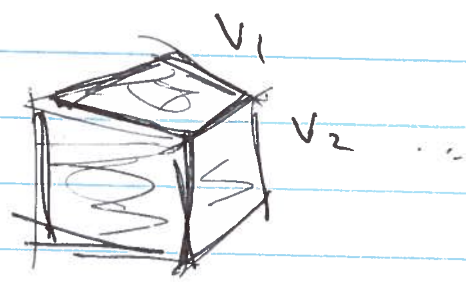
Quarter/eighth



$\frac{1}{2n}$.



Cube?



$$\Phi\big|_{center} = \frac{1}{4\pi}\oint d^2x' \, \Phi\big|_S \left(-\frac{\partial G}{\partial n'}\right)$$

$$= \sum_{i=1}^{N} V_i \left(-\frac{1}{4\pi}\oint_{S_i} d^2x' \, \frac{\partial G}{\partial n'}\right)$$

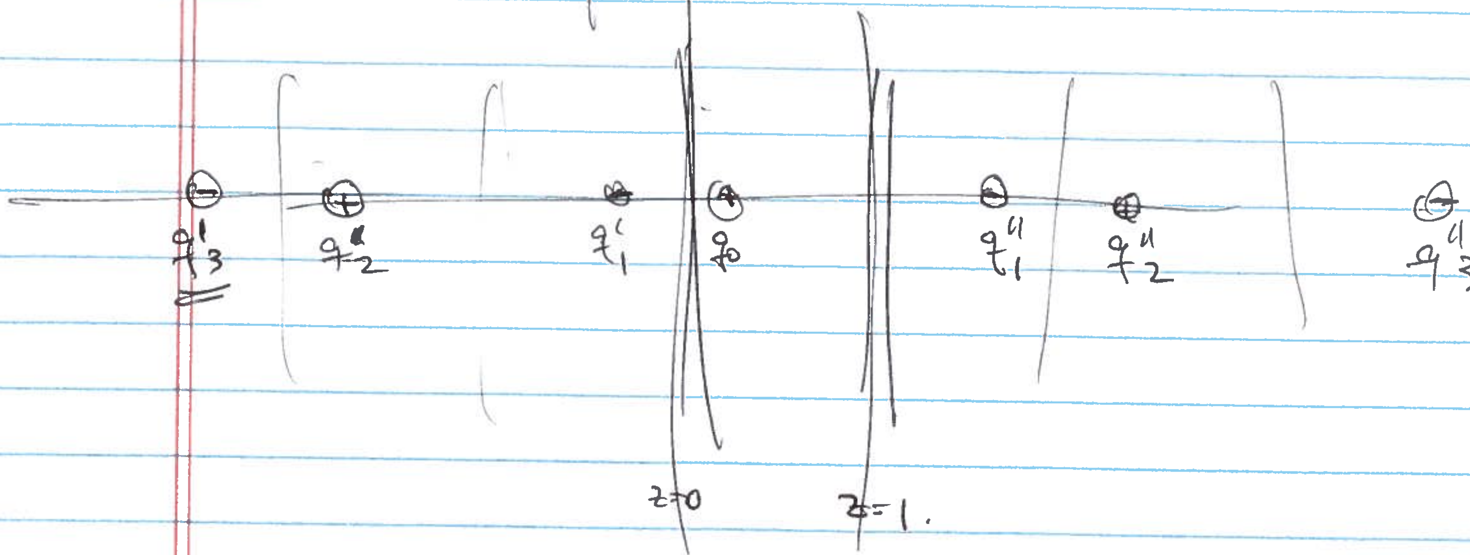
$$= k \cdot \left(\sum_{i=1}^{N} V_i\right) \qquad \Phi\big|_c = \frac{1}{6}\sum_{i=1}^{N} V_i$$

No max/min in interior ($\rho = 0$) →

$$V_0 = k\left(\sum_{i=1}^{N} V_0\right) = 6kV_0 \rightarrow k = \frac{1}{6}$$

G(curve)??

Between two planes



$$G = \sum_{k=1}^{\infty} \frac{q'_k}{|\vec{x}-\vec{x}'_k|} + \frac{q''_k}{|\vec{x}-\vec{x}''_k|} + \frac{q}{|\vec{x}-\vec{x}_0|}$$

x_k grows as k

$(-1)^k$ alternating } converges (conditionally)

Reliable method: Separation of Variables

Green's Function. $\nabla^2 G = -4\pi\,\delta^{(3)}(\vec{x}-\vec{x}')$

vanishes Almost Everywhere

$\Rightarrow$ look at Laplace's Equation $\nabla^2\Phi = 0$ { Boundary cond. / Green's function

Proc.: find general solution, match coefficients on S.

1st pass: Cartesian $\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

general approach. try. $\Phi = X(x)Y(y)Z(z)$.

$$\nabla^2\Phi = \frac{d^2X}{dx^2}\cdot Y\cdot Z + X\cdot\frac{d^2Y}{dy^2}\cdot Z + X\cdot Y\cdot\frac{d^2Z}{dz^2}.$$

$\div XYZ$ $$\frac{1}{X}X'' + \frac{1}{Y}Y'' + \frac{1}{Z}Z'' = 0$$

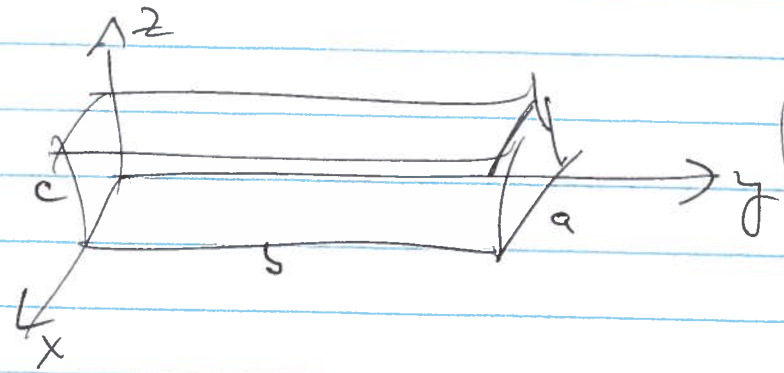
Each term must be constant.

$$\frac{1}{X}X'' = -\alpha^2 \qquad \frac{1}{Y}Y'' = -\beta^2 \qquad \frac{1}{Z}Z'' = +\gamma^2$$

$\gamma^2 = \alpha^2 + \beta^2$ (all possibilities)

$$\Phi = e^{\pm i\alpha x}\, e^{\pm i\beta y}\, e^{\pm \gamma z}$$ 8 terms (all products)

→ Need at least one of each sign.



$0 < x < a$
$0 < y < b$
$0 < z < c$

Choose: $\Phi = 0$ @ $x = 0$ $x = a$
$y = 0$ $y = a$.
$z = 0$

$\Phi(x, y, z = c) = V_s(x, y)$ ⟸ specified

$X = A_+ e^{i\alpha x} + A_- e^{-i\alpha x}$ $A_+ = -A_-$ $X = \sin \alpha x$

Similarly. $Y = \sin \beta y$ $Z = \sinh \gamma z$
$= \frac{1}{2}\left(e^{\gamma z} - e^{-\gamma z}\right)$

$X = a$ $\sin \alpha a = 0$ $\alpha_n a = n\pi$ $\alpha_n = \frac{n\pi}{a}$ $n = 1, \ldots$

$Y = \beta$ $\beta_m = \frac{m\pi}{a}$ $m = 1, \ldots$

$$\gamma_{nm} = \alpha_n^2 + \beta_m^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

$$\boxed{\Phi(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sinh \gamma_{nm} z}$$

other faces swap roles of (x, y, z)

top $\Leftrightarrow$ bottom ~~a~~ . $\sinh\gamma z \to \sinh\gamma(c-z)$

watch B.C.

$$\sum_{nm} A_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sinh\gamma_{nm} c = V_s(x,y)$$

2D Fourier sin series use orthogonality

$$\int_0^a dx \cdot \sin\frac{n\pi x}{a} \cdot \sin\frac{n'\pi x}{a} = ?? \qquad \frac{1}{2}\cdot \delta_{nn'} \cdot a.$$

$$\boxed{\langle \sin x\rangle = \langle \cos x\rangle = 0} \quad \boxed{\langle \sin^2 x\rangle = \langle \cos^2 x\rangle = \tfrac{1}{2}}$$

$\frac{1}{2}(1 \pm \cancel{\cos 2x})$

$$\cos(p-q) = \cos p \cos q + \sin p \sin q$$

$$\cos(p+q) = \cos p \cos q - \sin p \sin q.$$

$$\Rightarrow \frac{1}{2}\left[\cos(p-q) - \cos(p+q)\right] = \sin p \sin q$$