9/16/2015

$\nabla^2 \Phi = 0$ in ~~surface~~ block $0<x<a$ $0<y<b$, $0<z<c$

$\Phi = 0$ front, back, left, right, bottom

$$\rightarrow \Phi = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh(\gamma_{nm} z)$$

$$\gamma_{nm}^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

water. BC.

$$\Phi(\text{top}) = \sum_{nm} A_{nm} \cdots \underline{\sinh \gamma_{nm} c} = V_S(x,y).$$

$$\langle \sin x \rangle = 0 \qquad \langle \sin^2 x \rangle = \tfrac{1}{2}.$$

$$\cos(p-q) = \cos p \cdot \cos q + \sin p \sin q$$

$$\cos(p+q) = \cos p \cdot \cos q - \sin p \sin q$$

$$\tfrac{1}{2}\left(\cos(p-q) - \cos(p+q)\right) = \sin p \sin q.$$

$$\int_0^a dx \, \sin\frac{n\pi x}{a} \, \sin\frac{n'\pi x}{a}$$

$$= \int_0^a dx \, \frac{1}{2}\left[\cos\left(\frac{(n-n')\pi x}{a}\right) - \cos\left(\frac{(n+n')\pi x}{a}\right)\right]$$

$$= \frac{1}{2}\left[\frac{a}{\pi(n-n')} \cancel{\sin\left(\frac{(n-n')\pi x}{a}\right)} - \frac{a}{\pi(n+n')} \cancel{\sin\left(\frac{(n+n')\pi x}{a}\right)}\right]_0^a$$

$n \neq n'$

$n = n'$ $$\int_0^a dx \, \frac{1}{2}\left(\cos\left(\frac{(n-n')\pi x}{a}\right)\right) = \int_0^a \frac{1}{2} dx = \frac{1}{2} a.$$

$$\boxed{\int_0^a dx \, \sin\frac{n\pi x}{a} \, \sin\frac{n'\pi x}{a} = \frac{1}{2} \cdot a \cdot \delta_{nn'}}$$

$$\int_0^a dx \int_0^b dy \, \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) V_s(x,y)$$

$$= \int_0^a dx \int_0^b dy \sum_{n'm'} A_{n'm'} \sin\left(\frac{n'\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m'\pi y}{b}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh \gamma_{n'm'} c$$

$$= \sum_{n'm'} A_{n'm'} \cdot \frac{1}{2} a \, \delta_{n'n} \cdot \frac{1}{2} b \, \delta_{m'm} \cdot \sinh \gamma_{n'm'} c$$

$$= \underline{\frac{1}{4} ab \cdot A_{nm} \cdot \sinh \gamma_{nm} c}$$

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Case: $V_s(x,y) = V_0$ = constant.

$$(\text{LHS}) = V_0 \int_0^a dx \cdot \sin\frac{n\pi x}{a} \cdot \int_0^b dy \cdot \sin\frac{m\pi y}{b}$$

$$= V_0 \left(-\frac{a}{n\pi}\cos\frac{n\pi x}{a}\right)_0^a \left(-\frac{b}{m\pi}\cos\frac{m\pi y}{b}\right)_0^b$$

$$= +V_0 \frac{ab}{mn\pi^2}\Big(1-\underbrace{\cos n\pi}_{(-1)^n}\Big)\Big(1-\underbrace{\cos m\pi}_{(-1)^m}\Big)$$

$$\begin{cases} 2\ (\text{odd}) \\ 0\ (\text{even}) \end{cases} \qquad \begin{cases} 2\ (\text{odd}) \\ 0\ (\text{even}) \end{cases}$$

$$A_{nm} = \frac{\left(\frac{4V_0 ab}{mn\pi^2}\right)}{\left(\frac{1}{4} ab \sinh\gamma_{nm} c\right)} = \begin{cases} \frac{16V_0}{\pi^2 nm}\frac{1}{\sinh\gamma_{nm}c} & (\text{both odd}) \\ 0 & \text{otherwise} \end{cases}$$

$$\Phi = \sum_{\substack{j=0 \\ n=2j+1}}^{\infty} \sum_{\substack{k=0 \\ m=2k+1}}^{\infty} \frac{16V_0}{\pi^2 nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \times \frac{\sinh\gamma_{nm} z}{\sinh\gamma_{nm} c}$$

convergence

case $a=b=c$: $\gamma_{nm} = \sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{a}\right)^2} = \frac{\pi}{a}\sqrt{n^2+m^2} = \frac{N\pi}{a}$

$\sinh\gamma z \to \sinh N\pi z/a$

$\sinh\gamma c \to \sinh N\pi.$

4

$N\pi$ not small even for $n = m = 1$ $\to \sqrt{2}\pi = 4.44288$

$z/a \ll 1$. $\sinh(\gamma z) \simeq \gamma z$ $\Phi \to$ linearly to zero.

$N\pi$ big. $\sinh x \to \frac{1}{2}(e^x - e^{-x}) \to \frac{1}{2} e^x$

$$\frac{\sinh \gamma z}{\sinh \gamma z} \to \frac{\frac{1}{2} e^{\gamma z}}{\frac{1}{2} e^{\gamma c}} = e^{-\gamma(a-z)} = e^{-N\pi(1-\frac{z}{a})}$$

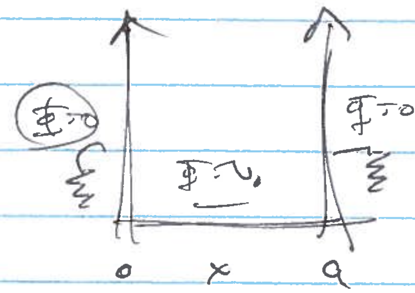
eventually. goes to zero exponentially.

$z=c$ doesn't converge as fast, but. B.C.!

#5

(2D) . x-y . (constant in z). $(Z(z)=1)$

$$\Phi = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{a} e^{-\frac{n\pi y}{a}}$$



$$= \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{4V_0}{\pi n} \left(\sin\frac{n\pi x}{a} e^{-\frac{n\pi y}{a}} \right)$$

$$\mathrm{Im}\left(e^{i\frac{n\pi x}{a}} e^{-\frac{n\pi y}{a}} \right) = \mathrm{Im}\left(e^{\frac{in\pi}{a}(x+iy)} \right)$$

$$= F(x+iy).$$

let $Z = e^{i\frac{n\pi}{a}(x+iy)}$.

$$\Phi = \mathrm{Im}\left[\frac{4V_0}{\pi} \mathrm{Im}\left(Z + \frac{1}{3}Z^3 + \frac{1}{5}Z^5 + \cdots \right) \right)$$

$$-\log(1-Z) = Z + \frac{1}{2}Z^2 + \frac{1}{3}Z^3 + \cdots$$

$$\log(1+Z) = Z - \frac{1}{2}Z^2 + \frac{1}{3}Z^3 + \cdots$$

$$\frac{1}{2}\log(1+Z) - \frac{1}{2}\log(1-Z) = Z + \frac{1}{3}Z^3 + \frac{1}{5}Z^5 + \cdots$$

$$\Phi = \frac{4V_0}{\pi} \mathrm{Im}\left[\frac{1}{2} \log\left(\frac{1+Z}{1-Z} \right) \right]$$

$$\frac{1+Z}{1-Z} = \frac{(1+Z)(1-Z^*)}{(1-Z)(1-Z^*)} = \frac{1+Z-Z^* - |Z|^2}{1-(Z+Z^*)+|Z|^2}$$

$$\frac{1-|z|^2 + z - z^*}{1+|z|^2 - (z+z^*)} = R\, e^{i\Theta}$$

$$\underline{\log = \log R + i\,\Theta}\,.$$

$$\tan\Theta = \frac{\operatorname{Im}\left(\frac{1+z}{1-z}\right)}{\operatorname{Re}\left(\frac{1+z}{1-z}\right)} = \frac{2\operatorname{Im} z}{1-|z|^2}$$

$$= \frac{2\cdot \sin\frac{\pi x}{a}\, e^{-\frac{\pi y}{a}}}{1 - e^{-\frac{2\pi y}{a}}} = \frac{\sin\frac{\pi x}{a}}{\sinh\frac{\pi y}{a}}.$$

$$\boxed{\Phi = \frac{2V_0}{\pi}\tan^{-1}\left(\frac{\sin\frac{\pi x}{a}}{\sinh\frac{\pi y}{a}}\right)}$$

$\underline{x=0}$, $\underline{x=a}$ $\quad \sin 0 = \sin\pi = 0$ ✓

$\underline{y\to 0}$ $\underline{x\neq 0}$ $\quad \frac{\sin\frac{\pi x}{a}}{\left(\frac{\pi y}{a}\right)} \to \infty \quad \tan^{-1} \to \frac{\pi}{2}$ ✓

$\nabla^2\Phi = 0$ ✓ $\quad$ (non trivial)