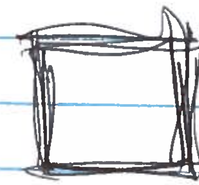
9/18/2015

Green's Function : 1st time: 2D

$0<x<1$
$0<y<1$



(1) $\nabla^2 \Phi = -4\pi \delta^{(3)}(\vec{x}-\vec{x}')$?

(2) $\Phi|_S = 0$

Need : expansion of $\delta(x-x')$

$$\boxed{\delta(x-x') = \sum_{n=1}^{\infty} D_n \cdot \sin\frac{n\pi x}{a}}$$

$$\int_0^a dx' \cdot \delta(x-x') \cdot \sin\frac{n\pi x}{a} = \sin\frac{n\pi x'}{a}$$

$$= \int_0^a dx \sum_{n'=1}^{\infty} D_n \sin\frac{n'\pi x}{a} \sin\frac{n'\pi x}{a}$$

$$= \sum_{n'=1}^{\infty} D_{n'} \cdot \left(\frac{1}{2} a \,\delta_{nn'}\right) = \frac{1}{2} a \, D_n$$

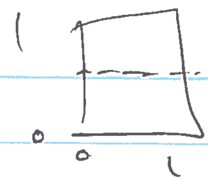
$$D_n = \frac{2}{a} \sin\frac{n\pi x'}{a}$$

$$\boxed{\delta(x-x') = \sum_{n=1}^{\infty} \frac{2}{a} \sin\frac{n\pi x}{a} \cdot \sin\frac{n'\pi x}{a}}$$

$\boxed{f = \sum C_n \sin\frac{n\pi x}{a}}$

$$\int_0^a dx' f(x') \sin(x-x') = \int_0^a dx \cancel{\sum_n C_n \sin\frac{n\pi x}{a} \sin}$$

$$= \int_0^a dx \left(\sum_n C_n \sin\frac{n\pi x'}{a}\right)\left(\sum_{n'} \frac{2}{a} \sin\frac{n'\pi x}{a} \sin\frac{n'\pi x'}{a}\right) = \sum_n C_n \sin\frac{n\pi x}{a}.$$



Green's Function

$$\nabla^2 G = -4\pi\, \delta^{(2)}(\vec{x}-\vec{x}') = -4\pi\, \delta(x-x')\,\delta(y-y') = 0 \text{ A.E.}$$

$G|_S = 0$

Try: $G = \sum_{n=1}^{\infty} C_n \underbrace{\sin n\pi x'}_{\checkmark}\ \underbrace{e^{\pm n\pi y'}}_{??}$ can't satisfy both $y'=0$ $y'=1$

Good Thing: Two regimes:

$y'=0$ ✓

$$G_<(x',y') = \sum_{n=1}^{\infty} C_n^< \sin n\pi x' \sinh n\pi y' \quad (y'<y)$$

$$G_>(x',y') = \sum_{n=1}^{\infty} C_n^> \sin n\pi x' \sinh n\pi(1-y') \quad (y'>y)$$

$y'=1$ ✓

join @ $y'=y$. smooth for $x' \neq x$.

$$\sum_{n=1}^{\infty} C_n^< \sin n\pi x' \sinh n\pi y = \sum_{n=1}^{\infty} C_n^> \sin n\pi x' \sinh n\pi(1-y)$$

term-by-term independent (in fact orthogonal)

$\Rightarrow$ each coefficient must match

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$$C_n^< \sinh n\pi y = C_n^> \sinh n\pi(1-y)$$

$$\frac{C_n^<}{\sinh n\pi(1-y)} = \frac{C_n^>}{\sinh n\pi y} = \underline{\underline{C_n}}.$$

$$G(x,y;x',y') = \sum_{n=1}^{\infty} C_n \cdot \sin n\pi x' \sinh n\pi y_< \sinh n\pi(1-y_>)$$

$\uparrow_x$

$\nabla'^2 = 0$ $x'=0$ $x'=1$ $y'=0$ $y'=1$.

$$\nabla'^2 G = \sum_{n=1}^{\infty} C_n \left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) \sin n\pi x' \sinh n\pi y_< \sinh n\pi(1-y_>)$$

$$= -4\pi \cdot \sum_{n=1}^{\infty} 2 \cdot \sin n\pi x' \sin n\pi x \quad \delta(y-y')$$

term-by-term:

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2\right) C_n \sinh n\pi y_< \sinh n\pi(1-y_>)$$

$$= -4\pi \cdot 2 \sin n\pi x \cdot \delta(y-y')$$

Integrate $\int_{y-\varepsilon}^{y+\varepsilon} dy'$

④

$$\int_{y-\varepsilon}^{y+\varepsilon} dy' \frac{\partial^2 g_n}{\partial y'^2} = \left.\frac{\partial g_n}{\partial y'}\right|_{y'=y+\varepsilon} - \left.\frac{\partial g_n}{\partial y'}\right|_{y'=y-\varepsilon}.$$

$$\int_{y-\varepsilon}^{y+\varepsilon} dy'\, n^2\pi^2 g_n \simeq n^2\pi^2 g_n(y'=y)\, 2\varepsilon \longrightarrow 0.$$

$$\int_{y-\varepsilon}^{y'+\varepsilon} dy'.\ (-8\pi \sin n\pi x.\ \delta(y-y')). = \underline{\underline{-8\pi \sin n\pi x}}$$

$$y' = y+\varepsilon = y_>$$

$$C_n \left[\frac{\partial}{\partial y'} \left(\sinh n\pi y.\ \sinh n\pi(1-y') \right) - \frac{\partial}{\partial y'} \left(\sinh n\pi y' \sinh n\pi(1-y) \right) \right]$$

$$y' = y - \varepsilon = y_<$$

$$= C_n \left[(-n\pi) \sinh n\pi y \cosh n\pi (1-y') - (n\pi) \cosh n\pi y' \sinh n\pi (1-y) \right]$$

$$= -n\pi C_n \left[\sinh n\pi y \cosh n\pi (1-y) + \cosh n\pi y . \sinh n\pi (1-y) \right)$$

$$\sinh (n\pi y + n\pi(1-y)) \quad \therefore \ \sinh n\pi$$

$$= -n\pi C_n \sinh n\pi = -8\pi \sin n\pi x$$

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$$C_n = \frac{8 \cdot \sin n\pi x}{n \cdot \sinh n\pi}$$

$$\boxed{G(x,y;x'y') = \sum_{n=1}^{\infty} \frac{8}{n \sinh n\pi} \sin n\pi x \sin n\pi x' \times \sinh n\pi y_< \sinh n\pi(1-y_>)}$$

$\nabla'^2 G = 0$ A.E. $\nabla^2 G = 0$ A.E.

$G|_{x=0} = G|_{x=1} = G|_{y=0} = G|_{y=1} \to 0$

$G(\vec{x},\vec{x}') = G(\vec{x}',\vec{x})$

$$\frac{\sinh n\pi y_< \sinh n\pi(1-y_>)}{\sinh n\pi} \to \frac{\frac{1}{2}e^{n\pi y_<} \frac{1}{2}e^{n\pi(1-y_>)}}{\frac{1}{2}e^{n\pi}}$$

$$\frac{1}{2}e^{-n\pi(y_> - y_<)}$$

$n\pi|y-y'|$

converges exponentially for $y \neq y'$.

$y_<, y_>$ typical