9/23/2015

$$\nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

$$\Phi = R(r)\, P(\theta)\, Q(\phi)$$

$$\rightarrow \underbrace{\frac{r^2}{R}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right)}_{+l(l+1)} + \underbrace{\frac{1}{P\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \frac{1}{\sin^2\theta}\boxed{\frac{1}{Q}\frac{d^2Q}{d\phi^2}}}_{-l(l+1)} = 0$$

oscillator

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -m^2$$

$$\boxed{Q = e^{im\phi}}$$

m = integer

$$\frac{1}{P}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -l(l+1)$$

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) = +l(l+1)$$

$$\boxed{r^2 R'' + 2rR' - l(l+1)R = 0}$$

homogeneous in powers of r $\rightarrow R \sim r^p$

$$p(p-1) + 2p - l(l+1) = 0$$

$$p(p+1) = l(l+1)$$

$$p = \begin{cases} l \\ -(l+1) \end{cases}$$

$$\boxed{R = r^l,\ r^{-(l+1)}}$$

②

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]P=0.$$

let $x=\cos\theta$ $\qquad \frac{dP}{d\theta}=\frac{dP}{dx}\cdot\frac{dx}{d\theta}=-\sin\theta\frac{dP}{dx}$

$$\frac{dP}{dx}=-\frac{1}{\sin\theta}\frac{dP}{d\theta}.$$

$$\boxed{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P=0}$$

(generalized) Legendre equation $\rightarrow P_\ell^m(x)$

"associated Legendre function"

Start with $m=0$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\ell(\ell+1)P=0$$

O.D.E: $\qquad P(x)=\sum_{j=0}^{\infty}a_j\,x^{j+\alpha}$ $\qquad$ (α allows $\sqrt{x}$, etc.)

$$\frac{d^2P}{dx^2}=\sum_j(\alpha+j)(\alpha+j-1)\,a_j\,x^{\alpha+j-2}$$

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$$(1-x^2)P'' - 2xP' + \ell(\ell+1)P$$

$$= \sum_{j=0}^{\infty} \Big[(1-x^2)(j+\alpha)(j+\alpha-1)\, a_j\, x^{j+\alpha-2} - 2x(j+\alpha)\, a_j\, x^{j+\alpha-1} + \ell(\ell+1)\, a_j\, x^{j+\alpha-2} \Big]$$

term: $\underline{(j+\alpha)(j+\alpha-1)\, x^{j+\alpha-2}}$ (no x^2 ...)

let $j+\alpha-2 = j'+\alpha$ $\quad j' = j-2 \quad j = j'+2$

$j=0 \rightarrow j'=-2$

$$\text{PEq.} = \alpha(\alpha-1)\, a_0\, x^{\alpha-2} + \alpha(\alpha+1)\, a_1\, x^{\alpha-1}$$

$j'=-2$
$j'=-1$

$$+ \sum_{j'=0} \Big[(j'+\alpha+2)(j'+\alpha+1)\, a_{j'+2}\, x^{j'+\alpha} - \big[(j+\alpha)(j+\alpha-1) + 2(j+\alpha) - \ell(\ell+1)\big] a_j\, x^{j+\alpha} \Big]$$

Vanish (for any x) $\rightarrow$ $\alpha(\alpha-1)\, a_0 = 0$ $\quad j=-2$

$\Rightarrow$ $\alpha(\alpha+1)\, a_1 = 0$ $\quad j=-1$

others $\rightarrow$ $a_{j+2} = \dfrac{(j+\alpha)(j+\alpha+1) - \ell(\ell+1)}{(j+\alpha+1)(j+\alpha+2)}\, a_j$

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if $a_0 \neq 0$ $\quad \alpha = 0$ or $\alpha = 1$

$\alpha = 0 \quad a_0 + a_2 x^2 + a_4 x^2 + \cdots$

$\alpha = 1 \quad a_0 x + a_2 x^3 + a_4 x^5 + \cdots$

if $a_1 \neq 0$ $\quad \alpha = 0$ or $\alpha = -1$

$\alpha = 0 \quad a_1 x + a_3 x^3 + a_5 x^5 + \cdots$

$\alpha = -1 \quad a_1 + a_3 x^2 + a_5 x^2 + \cdots$

same series.

recursion $j + \alpha \to j$

choose $\alpha = 0$ solution (don't lose anything).
[could have left off α, but practice for later)

$\alpha = 0 \qquad a_{j+2} = \dfrac{j(j+1) - l(l+1)}{(j+1)(j+2)} \, a_j$

$j \to$ large $\qquad \dfrac{a_{j+2} x^{j+2}}{a_j x^j} \to \left(\dfrac{a_{j+2}}{a_j}\right) x^2 \Rightarrow x^2$

series converges for $|x| < 1$.

$x = \pm 1 \quad$ ($\pm$ z-axis?) $\cos\theta = \pm 1$ $\qquad$ diverges logarithmically unless truncates.

l = integer $\qquad j = l \qquad a_{j+2} = \cancel{j(j+1) - l(l+1)}\, a_j \quad 0$

odd l $\to$ odd series terminates
even l $\to$ even series terminates

$\ell = 0$ odd series $\rightarrow$ $\boxed{Q_0(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)}$

even "series": $a_0 \rightarrow a_2 = \frac{(0)(1) - (0)(1)}{(1)(2)} = 0$.

$P_0 = a_0$ normalization condition $P_\ell(1) = 1$

$\rightarrow a_0 = 1$ $\boxed{P_0(x) = 1}$

$\ell = 1$ $a_1 \longrightarrow a_3 = 0$ etc.

$P_1(x) = a_1 x$ $P_1(1) = a_1 = 1$ $\boxed{P_1(x) = x}$

$(j+2)$ j $j+1$ ℓ $\ell+1$

$\ell = 2$ $a_0 \longrightarrow a_2 = \frac{(0)(1) - (2)(3)}{(1)(2)} = -3a_0$

a_4 etc $= 0$

$P_2 = a_0 - 3a_0 x^2$

$P_2(1) = a_0 - 3a_0 = -2a_0 = 1$ $\boxed{P_2 = \frac{3}{2}x^2 - \frac{1}{2}}$

$a_0 = -\frac{1}{2}$

$\boxed{P_3 = \frac{5}{2}x^3 - \frac{3}{2}x}$

$\boxed{P_4 = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}}$

Rodrigues' formula:

$$P_\ell = \sum_{k=0}^{[\ell/2]} (-1)^k \frac{(2\ell-2k)!}{2^\ell k!\,(\ell-k)!\,(\ell-2k)!} x^{\ell-2k}$$

$j = \ell - 2k$ $\quad$ $k=0 \to x^\ell$. P_ℓ has degree $= \ell$. ✓

even $\ell - 2[\ell/2] = 0$

odd $\ell - 2[\ell/2] = 1$. remainder.

$$\frac{a_{j+2}}{a_j} = \frac{a(k-1)}{a(k)} = \frac{\dfrac{(-1)^{k-1}(2\ell-2k+2)!}{\cancel{2^\ell}(k-1)!\,(\ell-k+1)!\,(\ell-2k+2)!}}{\dfrac{(-1)^k (2\ell-2k)!}{\cancel{2^\ell}\,k!\,(\ell-k)!\,(\ell-2k)!}}$$

$$= (-1)\frac{(2\ell-2k+2)(2\ell-2k+1)\cdot k}{(\ell-k+1)(\ell-2k+2)(\ell-2k+1)}$$

$$= (-)\frac{\cancel{2}\,(2\ell-(\ell-j)+1)\left(\frac{\ell-j}{\cancel{2}}\right)}{(\cancel{\ell}-\cancel{(\ell}-j)+2)(\cancel{\ell}-\cancel{(\ell}-j)+1)}$$

$$= (-)\frac{(\ell+j+1)(\ell-j)}{(j+2)(j+1)} = (-)\frac{(\ell^2-j^2+\ell-j)}{(j+2)(j+1)}$$ ✓