9|25|205

Legendre. $\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P = 0$.

series → ℓ = integer. $\sum_{j=0}^{\infty} a_j x^j$

$$\boxed{a_{j+2} = \frac{j(j+1) - \ell(\ell+1)}{(j+1)(j+2)}.}$$ (3.14)

($\ell=0$)

$P_\ell(1) = 1$ → specified.

Somebody.

$$P_\ell = \sum_{k=0}^{[\ell/2]} \frac{(-1)^k}{2^\ell k!(\ell-k)!} \frac{(2\ell-2k)!}{(\ell-2k)!} x^{\ell-2k}.$$

(verified.) ✓

$$\frac{d^l}{dx^l}\left(x^{2l-2k}\right) = \underbrace{(2l-2k)(2l-2k-1)\cdots(l-2k+1)}_{\frac{(2l-2k)!}{(l-2k)!}}\, x^{l-2k}$$

$$P_l = \sum_{k=0}^{[l/2]} \frac{1}{2^l}\frac{1}{l!}\cdot\frac{l!}{k!(l-k)!}(-1)^k \frac{d^l}{dx^l} x^{2l-2k}$$

$$= \frac{1}{2^l}\frac{1}{l!}\frac{d^l}{dx^l}\underbrace{\sum_{k=0}^{l}(-1)^k\frac{l!}{k!(l-k)!}x^{2(l-k)}}_{(x^2-1)^l}$$

(Bigger k's $\to 0$)

(3.16)

$$\boxed{P_l = \frac{1}{2^l}\frac{1}{l!}\frac{d^l}{dx^l}(x^2-1)^l}$$

Rodrigues' Formula.

Normalization, let $x = 1-\varepsilon$

$$(x^2-1) = \cancel{1}-2\varepsilon+\varepsilon^2-\cancel{1} = -2\varepsilon+\varepsilon^2 = -2\varepsilon\left(1-\frac{\varepsilon}{2}\right)$$

$$P_l(1-\varepsilon) = \frac{1}{2^l}\frac{1}{l!}\left(-\frac{d}{d\varepsilon}\right)^l(-1)^l(2\varepsilon)^l\left(1-\frac{\varepsilon}{2}\right)^l$$

$$= \frac{1}{l!}\frac{d^l}{d\varepsilon^l}\left(\varepsilon^l - \frac{l}{2}\varepsilon^{l+1}\right)$$

Stopping.

$$= \frac{1}{l!}\left(l! - \frac{l}{2}(l+1)!\,\varepsilon\right) = 1 - \frac{l(l+1)}{2}\varepsilon$$

$(x^2-1)^l$: polynomial of degree $2l$

$\left(\frac{d}{dx}\right)^l \rightarrow$ polynomial of degree $= l$

parity $= (-1)^l$ orthogonal

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_l}{dx}\right] + l(l+1)P_l = 0$$

$$\int_{-1}^{1} dx\, P_{l'}(x)\left[\frac{d}{dx}\left((1-x^2)\frac{dP_l}{dx}\right) + l(l+1)P_l\right] = 0$$

parts.

$$\left[P_{l'}(x)\underbrace{(1-x^2)}_{=0}\frac{dP_l}{dx}\right]_{-1}^{1} - \int_{-1}^{1} dx\,\frac{dP_{l'}}{dx}(1-x^2)\frac{dP_l}{dx} + \int_{-1}^{1} dx\, l(l+1)P_{l'}P_l$$

parts again.

or. $l \leftrightarrow l'$, subtract.

$$[l(l+1) - l'(l'+1)]\int_{-1}^{1} dx\, P_{l'}(x)P_l(x) = 0$$

$l \neq l'$ $\boxed{\int_{-1}^{1} dx\, P_{l'}(x)P_l(x) = 0}$

orthogonal

4

Normalization $\int_{-1}^{1} dx\, P_\ell^2(x) = ??$

$\ell=0$ $\int_{-1}^{1} dx\, P_0(x) = \int_{-1}^{1} dx \cdot 1 = 2$

$\ell=1$ $\int_{-1}^{1} dx\, (x)^2 = \left(\frac{x^3}{3}\right)_{-1}^{1} = 2/3$

$= \frac{9}{10} - \frac{10}{10} + \frac{5}{10}$

$\ell=2$ $\int_{-1}^{1} dx \left[\frac{9}{4}x^4 - 2\cdot\frac{3}{2}\cdot\frac{1}{2}x^2 + \frac{1}{4}\right] = \frac{9}{4}\cdot\frac{2}{5} - \frac{3}{2}\left(\frac{2}{3}\right) + \frac{2}{4} = \frac{4}{10}$

$= 2/5$

Extrapolate: $\frac{2}{2\ell+1}$

Prove:

Rodrigues' $\rightarrow$ $N_\ell = \int_{-1}^{1} dx \left[\frac{1}{2^\ell}\frac{1}{\ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\right]\left[\frac{1}{2^\ell}\frac{1}{\ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell\right]$

parts ℓ times $\rightarrow$ $\left(\frac{1}{2^\ell}\frac{1}{\ell!}\right)^2 (-1)^\ell \int_{-1}^{1} dx\, \frac{d^{2\ell}}{dx^{2\ell}}(x^2-1)^\ell \; (x^2-1)^\ell$

$$\boxed{N_\ell = \frac{(2\ell)!}{(2^\ell \ell!)^2} \int_{-1}^{1} dx\, (1-x^2)^\ell}$$

$x = \cos\theta$. $\int_0^{\pi} \sin^{2\ell+1}\theta\, d\theta$.

JDJ: induction on ℓ.

Beta function. $\boxed{B(p,q) = \int_0^1 dt\, t^{p-1} (1-t)^{q-1}}$

$t = x^2$ $\quad dt = 2x\,dx$ $\quad B(p,q) = \int_0^1 2x\,dx\, (x^2)^{p-1} (1-x^2)^{q-1}$

$x = \sin\theta$ $\quad dx = \cos\theta\, d\theta$ $\quad \boxed{B(p,q) = 2\int_0^{\pi/2} d\theta\, (\sin\theta)^{2p-1} (\cos\theta)^{2q-1}}$

$$\int_0^{\pi} d\theta\, (\sin\theta)^{2l+1} = 2\int_0^{\pi/2} d\theta\, (\sin\theta)^{2l+1}$$

$2l+1 = 2p-1$

$0 = 2q-1$

$$\boxed{N_l = B\left(l+1, \tfrac{1}{2}\right)}$$

$p = l+1$

$q = 1/2$

---

relate to $\Gamma(p)$, $\Gamma(q)$'s

$$\Gamma(p) = \int_0^\infty t^{p-1} dt\, e^{-t} = \int_0^\infty \left(\tfrac{1}{2}x^2\right)^{p-1} \cdot x\,dx\, e^{-\frac{1}{2}x^2}$$

$$\Gamma(p)\Gamma(q) = \int_0^\infty \left(\tfrac{1}{2}\right)^{p-1} x^{2p-1} dx\, e^{-x^2/2} \int_0^\infty \left(\tfrac{1}{2}\right)^{q-1} y^{2q-1} dy\, e^{-y^2/2}$$

$$= \left(\tfrac{1}{2}\right)^{p+q-2} \int_0^\infty r\,dr \int_0^{2\pi} d\theta\, r^{2p+2q-2} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1}$$

$$= \Gamma(p+q) \cdot 2\int_0^{\pi/2} (\sin\theta)^{2p-1} (\cos\theta)^{2q-1} d\theta$$

6

$$\boxed{B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}}$$

<u>wes</u>

$$B(l+1,\tfrac{1}{2}) = \frac{\Gamma(l+1)\Gamma(\tfrac{1}{2})}{\Gamma(l+\tfrac{3}{2})}$$

$\Gamma(p+1) = p\Gamma(p)$ etc. $\boxed{\Gamma(l+1) = l!}$

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} dt\, e^{-t} = \int_0^\infty \left(\frac{x^2}{2}\right)^{-\frac{1}{2}} x\,dx\, e^{-\frac{1}{2}x^2}$$

$$= \sqrt{2}\int_0^\infty dx\, e^{-\frac{1}{2}x^2} = \sqrt{2}\cdot\frac{1}{2}\sqrt{2\pi} = \sqrt{\pi}$$

$$\Gamma(l+\tfrac{3}{2}) = (l+\tfrac{1}{2})\Gamma(l+\tfrac{1}{2}) = (l+\tfrac{1}{2})(l-\tfrac{1}{2})\cdots\tfrac{1}{2}\Gamma(\tfrac{1}{2})$$

$$= \frac{(2l+1)(2l-1)\cdots 1}{2\cdot 2\cdots 2}\Gamma(\tfrac{1}{2}) = \frac{(2l+1)!!}{(2)^{l+1}}\sqrt{\pi}$$

$$\cancel{B(l+1,\tfrac{1}{2}) = \frac{(2l+1)!!\sqrt{\pi}}{(2)^{l+1}}}$$

$$B(l+1,\tfrac{1}{2}) = \frac{(l!)(\cancel{\sqrt{\pi}})}{\left(\frac{(2l+1)!!}{(2)^{l+1}}\cancel{\sqrt{\pi}}\right)} = \frac{2^{l+1}\cdot l!}{(2l+1)!!}$$

$$= \frac{2\,(2l)!!}{(2l+1)!!}$$

$\downarrow$ all the odds $\times$ all evens

$$\int_{-1}^{1} dx\, P_\ell^2(x) = \frac{(2\ell)!}{[2^\ell \cdot \ell!]^2} \; \frac{2\,(2\ell)!!}{(2\ell+1)!!}$$

$$= \frac{\cancel{(2\ell)!!}\,(2\ell-1)!!}{[\cancel{(2\ell)!!}]^2} \; \frac{2 \cdot \cancel{(2\ell)!!}}{(2\ell+1)!!} = \frac{2}{2\ell+1}$$

$$\boxed{\int_{-1}^{1} dx\, P_{\ell'}(x)\, P_\ell(x) = \frac{2}{2\ell+1}\,\delta_{\ell,\ell'}} \qquad (3.21)$$

Recursion "straightforward" (3.28)

$$\frac{dP_{\ell+1}}{dx} - \frac{dP_{\ell-1}}{dx} = (2\ell+1)\,P_\ell(x)$$

$$(\ell+1)\,P_{\ell+1}(x) - (2\ell+1)\,x\,P_\ell + \ell\,P_{\ell-1} = 0 \qquad 3.29 \; (a)$$

$$P_{\ell+1} = \frac{(2\ell+1)\,x\,P_\ell - \ell\,P_{\ell-1}}{\ell+1}$$