9/28/2015.

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_\ell}{dx}\right] + \ell(\ell+1)\, P_\ell(x) = 0$$

$$\boxed{P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell} \quad (3.16)$$

$P_\ell(1) = 1$

$P_\ell(-1) = (-1)^\ell$

$$\boxed{\int_{-1}^{1} dx\, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'}} \quad (3.21)$$

$$\boxed{\frac{dP_{\ell+1}}{dx} - \frac{dP_{\ell-1}}{dx} = (2\ell+1) P_\ell(x)} \quad (3.28)$$

$$\boxed{(\ell+1)P_{\ell+1} - (2\ell+1)\, x\, P_\ell + \ell P_{\ell-1} = 0} \quad (3.29)(b).$$

$P_0 = 1$ $(\ell = 0)$ $\quad P_1 = \dfrac{(1)(x)(1) - (0)(\quad)}{(1)} = x$

$(\ell = 1)$ $\quad P_2 = \dfrac{(3)(x)(x) - (1)(1)}{(2)} = \frac{3}{2}x^2 - \frac{1}{2}$

$$\boxed{P_{\ell+1} = \frac{(2\ell+1)\, x\, P_\ell - \ell P_{\ell-1}}{\ell+1}}$$

$|x| \le 1$ $\quad |P| \le 1$

stable, efficient.

$\rightarrow$ plot

$\underline{P_\ell(0)}$:

$$\cancel{(\ell+1)}\,P_{\ell+1} = \frac{\cancel{(2\ell+1)x\,P_\ell} - \ell\,P_{\ell-1}}{\ell+1} = -\frac{\ell}{\ell+1}\,P_{\ell-1}$$

$$P_0(0) = 1$$

$\ell=1$ $\quad P_2(0) = -\frac{1}{2}\,P_0 = -\frac{1}{2}$

$\ell=3$ $\quad P_4 = -\frac{3}{4}\,P_2 = +\frac{1\cdot 3}{2\cdot 4}$

$\ell=5$ $\quad P_6 = -\frac{5}{6}\,P_4 = -\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} = -\frac{3}{8}$

$$\boxed{P_{2k}(0) = (-1)^k\,\frac{(2k-1)!!}{(2k)!!}}$$

Complete set $(-1,1)$.

$$f(x) = \sum_{\ell=0}^{\infty} a_\ell \cdot P_\ell(x).$$

$$\int_{-1}^{1} dx \cdot f(x)\,P_\ell(x) = \sum_{\ell'=0}^{\infty} a_{\ell'} \cdot \delta_{\ell\ell'}\,\frac{2}{2\ell+1} = \frac{2a_\ell}{2\ell+1}$$

<u>let</u> $f(x) = \begin{cases} +1 & x > 0 \\ -1 & x < 0. \end{cases}$

$\ell = \text{even}$: $\int_{-1}^{1} dx\, f(x) P_\ell(x) = \int_0^1 dx\, P_\ell(x) + \int_{-1}^{0} dx (-) P_\ell(x)$

$-\int_0^1 dx' P_\ell(x') \qquad = .$

$x' = -x$

$\ell = \text{odd}$. $\int_{-1}^{1} dx\, f(x) dx = \int_0^1 dx\, P_\ell(x) - \int_{-1}^{0} dx\, P_\ell(x)$

$= 2 \int_0^1 dx\, P_\ell(x).$

<u>use</u>. $\frac{dP_{\ell+1}}{dx} - \frac{dP_{\ell-1}}{dx} = (2\ell+1) P_\ell.$ (3.28).

$$2\int_0^1 dx\, P_\ell(x) = \frac{2}{2\ell+1}\left[\int_0^1 dx \left(\frac{dP_{\ell+1}}{dx} - \frac{dP_{\ell-1}}{dx}\right)\right]$$

$$= \frac{2}{2\ell+1}\left[P_{\ell+1}(x) - P_{\ell-1}(x)\right]_0^1$$

$$= \frac{2}{2\ell+1}\left[\underbrace{\cancel{P_{\ell+1}(1)}}_{(1)} - P_{\ell+1}(0) - \underbrace{\cancel{P_{\ell-1}(1)}}_{(1)} + P_{\ell-1}(0)\right]$$

$$= \frac{2}{2\ell+1}\left[P_{2k}(0) - P_{2k+2}(0)\right]$$

$$= \frac{2}{2\ell+1}\left[\frac{(-1)^k (2k-1)!!}{(2k)!!} - (-1)^{k+1}\frac{(2k+1)!!}{(2k+2)!!}\right]$$

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$$\int_{-1}^{1} dx\, f(x) P_l = \frac{2}{2l+1} (-1)^k \frac{(2k-1)!!}{(2k+1)!!}\left(1+(+)\frac{2k+1}{2k+2}\right) = \frac{2}{2l+1} a_l$$

$$\boxed{a_l = (-1)^k \frac{(2k-1)!!}{(2k+2)!!}\left(2(2k+1)+1\right)} \qquad 2l+1 \qquad l = 2k+1$$

JDJ: $$\left(-\tfrac{1}{2}\right)^{\frac{l-1}{2}} \frac{(2l+1)\,(l-2)!!}{2\left(\frac{l+1}{2}\right)!}$$

$l=1$ $k=0$: $$(-1)^0 \frac{(-1)!!}{(2)!!}(3) = (+)(1)\left(\frac{3}{2}\right) = \frac{3}{2}$$

$l=3$ $k=1$: $$(-1)^1 \frac{(1)!!}{(4)!!}(7) = \frac{(-)(1)(7)}{(4)(2)} = -\frac{7}{8}$$

$l=5$ $k=2$: $$(-1)^2 \frac{(3)!!}{(6)!!}(11) = \frac{(+)(\cancel{3})(11)}{2\cdot 4\cdot \cancel{6}_{2}} = \frac{11}{16}$$

$l=7$ $k=3$: $$(-1)^3 \frac{(5)!!\,(15)}{(8)!!} = \frac{-\cancel{3}\cdot 5\cdot 15}{2\cdot 4\cdot \cancel{6}_{2}\cdot 8} = -\frac{75}{128}$$

Plot

Physics $\nabla^2 \Phi = 0$

$$\Phi = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

inside $r \to 0$ $B_\ell = 0$

outside $r \to \infty$ $A_\ell = 0$

$r = a$ $\Phi\Big|_S = \cancel{\sum c_\ell P_\ell(\cos\theta)} = \boxed{V_s(\theta) = \sum V_0 \cdot a_\ell \cdot P_\ell(\cos\theta)}$

$$\Phi_{out} = V_0 \left[\frac{3}{2}\frac{a^2}{r^2} P_1 - \frac{7}{8}\frac{a^4}{r^4} P_3 + \frac{11}{16}\frac{a^6}{r^6} P_5 - \frac{75}{128}\frac{a^8}{r^8} P_7 \right.$$
$$\left. r \cdots \right)$$

§2.7

$$\Phi_{axis} = V_0 \left[1 - \frac{(r^2 - a^2)}{r\sqrt{r^2 + a^2}} \right)$$

$$= V_0 \left(\frac{3}{2}\frac{a^2}{r^2} - \frac{7}{8}\frac{a^4}{r^4} + \frac{11}{16}\frac{a^6}{r^6} - \frac{75}{128}\frac{a^8}{r^8} \cdots \right)$$

(6)

$\boxed{\text{This is a valid tool.}}$

① $\longrightarrow$ Find Φ on-axis (symmetry).

② $\longrightarrow$ Expand. $\sum A_\ell r^\ell$ (inside) $\sum \frac{B_\ell}{r^{\ell+1}}$ (outside).

③ $\longrightarrow \sum A_\ell P_\ell(\cos\theta)$, $\sum \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$

Point charge let q' live at r', ($\theta' = 0$)
(choose axis through q').

For $r, \theta = 0)$ $\Phi = \left(\frac{q'}{4\pi\varepsilon_0}\right) \frac{1}{|\vec{x} - \vec{x}'|} = \left(\frac{q'}{4\pi\varepsilon_0}\right) \frac{1}{|\mathbf{r} - \mathbf{r}'|}$

$r > r'$

$$\Phi = \left(\frac{q'}{4\pi\varepsilon_0}\right)\left(\frac{1}{r}\right)\left(\frac{1}{1 - r'/r}\right)$$

$$= \left(\frac{q'}{4\pi\varepsilon_0}\right)\left(\frac{1}{r}\right)\left(1 + \frac{r'}{r} + \frac{r'^2}{r^2} \cdots\right)$$

$$\Phi = \left(\frac{q'}{4\pi\varepsilon_0}\right) \sum_{\ell=0}^{\infty} \frac{r'^\ell}{r^{\ell+1}}. \qquad (r' < r).$$

$r' > r$ $\quad r \leftrightarrows r'$.

$$\Phi = \left(\frac{q'}{4\pi\varepsilon_0}\right) \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}}.$$