9/30/2015. $\nabla^2 \Phi = 0$. $V_s(\theta)$: $\sum_{l=0}^{\infty} C_l P_l(\cos\theta)$

$\longrightarrow \Phi = \sum_{l=0}^{\infty} C_l \left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta)$ $\quad r > a$

$\left(\frac{r}{a}\right)^l$. $\quad r < a$.

$\Phi(\text{axis}) = \sum \frac{B_l}{r^{l+1}} \longrightarrow \Phi(r,\theta) = \sum \frac{B_l}{r^l} P_l(\cos\theta)$

Use: point charge q' @ $\vec{x}' = (r', \theta = 0)$.

know: $\Phi = \frac{q'}{4\pi\varepsilon_0} \frac{1}{|\vec{x} - \vec{x}'|}$

$\vec{x}$ also on-axis $\rightarrow \Phi = \frac{q'}{4\pi\varepsilon_0} \frac{1}{|r' - r|}$

$r > r'$ $\quad \Phi = \frac{q'}{4\pi\varepsilon_0} \cdot \frac{1}{r} \left(\frac{1}{1 - r'/r}\right) = \frac{1}{r}\left(1 + \frac{r'}{r} + \left(\frac{r'}{r}\right)^2 + \cdots\right)$

$= \frac{q'}{4\pi\varepsilon_0}\left(\frac{1}{r} + \frac{r'}{r^2} + \frac{r'^2}{r^3} + \cdots\right)$

$\Phi_{\text{axis}} = \frac{q'}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}}$.

$r' > r$: exchange roles of r, r'.

2

$$\Phi = \frac{q'}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}}$$

$\boxed{\text{off-axis}}$
$$\Phi = \frac{q'}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} r'^l \left(\frac{P_l(\cos\theta)}{r^{l+1}} \right).$$

$$\Phi = \sum_{l=0}^{\infty} \frac{q'}{4\pi\varepsilon_0} \cdot \frac{r_<^l}{r_>^{l+1}} P_l(\cos\theta).$$

geometrically θ = angle between $\hat{r}, \hat{r}'$

$\longrightarrow \hat{r} \cdot \hat{r}' = \cos\gamma$

$$\boxed{\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)} \quad (3.38)$$

Skip "conical Hole".

3

$m \neq 0$. $\Phi = R(r)\,P(\theta)\,Q(\phi)$.

$\uparrow r^{\ell}, \frac{1}{r^{\ell+1}}$. $\quad \nwarrow e^{im\phi}$.

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0.$$

"easy to verify". ✓ "Associated Legendre Function"

$$\boxed{P_\ell^m = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)}$$

Rodrigues $\rightarrow$ $P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{1}{2^\ell}\frac{1}{\ell!}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$

works for. $-\ell \le m \le \ell$.

$m = +\ell$ $\quad P_\ell^\ell = (-1)^\ell (1-x^2)^{\ell/2} \frac{1}{2^\ell}\frac{1}{\ell!}(2\ell)!$

$m = -\ell$ $\quad P_\ell^{-\ell} = (-1)^\ell (1-x^2)^{-\ell/2} \frac{1}{2^\ell}\frac{1}{\ell!}(x^2-1)^\ell$

$= \frac{1}{2^\ell}\frac{1}{\ell!}(1-x^2)^{\ell/2}$

$-\ell \le m \le \ell$. $\quad \boxed{P_\ell^{-m} = \frac{(\ell-m)!}{(\ell+m)!}(-1)^m P_\ell^m}$ not new.

$$P_1^1 = (-)(1-x^2)^{1/2} \frac{d}{dx}(P_1) = -\sqrt{1-x^2}$$

$$P_2^1 = (-)(1-x^2)^{1/2} \frac{d}{dx}(P_2) = -3x\sqrt{1-x^2}$$

$$P_2^2 = (-)^2(1-x^2) \frac{d^2}{dx^2}(P_2) = 3(1-x^2).$$

Integrate diff. eq. (same m).

$$\int_{-1}^{1} dx\, P_\ell^m(x)\, P_{\ell'}^m(x) = \frac{2}{2\ell+1} \frac{(\ell+m)!}{(\ell-m)!}\, \delta_{\ell,\ell'}$$

All problems with spherical geometry.

$$\boxed{Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)\, e^{im\phi}}$$

"spherical harmonics" ("Tesseral harmonics"

Orthonormal $\int d\Omega\, Y_{\ell m}^*(\theta,\phi)\, Y_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'}\,\delta_{mm'}$

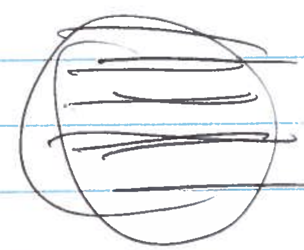
$$\boxed{Y_{\ell,-m} = (-1)^m Y_{\ell m}^*}$$

Y_{00} = constant $\boxed{Y_{00} = \frac{1}{\sqrt{4\pi}}}$

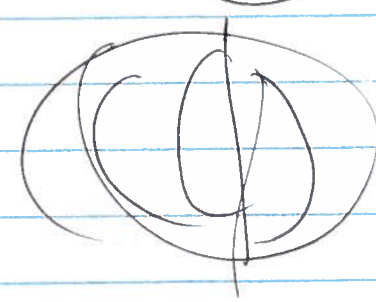
$Y_{10} \sim P_1 \sim \cos\theta$ $\boxed{Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta}$ $\sqrt{\frac{2l+1}{4\pi}} P_l$.

$Y_{11} \sim \underline{\sin\theta}\, e^{\pm i\phi}$ $\boxed{Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\phi}}$

$Y_{l,0} \sim P_l(\cos\theta)$ slices in θ $\underline{l}$ times (latitude)



$e^{im\phi}$ slices in ϕ. m times (longitude)



Complete. $f(\theta,\phi) = \sum A_{lm} Y_{lm}(\theta,\phi)$.

$\delta(\Omega - \Omega') = \underline{\delta(\cos\theta - \cos\theta')\, \delta(\phi-\phi')} = \frac{1}{\sin\theta}\delta(\theta-\theta')\delta(\phi-\phi')$

$= \sum_{lm} B_{lm} Y_{lm}(\theta,\phi)$

$\int d\Omega\, Y^*_{lm}(\theta,\phi)\, \delta(\Omega-\Omega') = Y^*_{lm}(\theta',\phi') = \int d\Omega'\, Y^*_{lm} \sum B_{lm} Y_{lm}$ $= B_{lm}$

$\delta(\Omega - \Omega') = \sum_{l,m} Y^*_{lm}(\theta',\phi')\, Y_{lm}(\theta,\phi)$.

$|lm\rangle\langle lm|$.