10/2/2015

$$\boxed{Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi}}$$

$$Y_{lm} \sim (\sin\theta)^m \left(\frac{d^m}{dx^m} P_l(\cos\theta)\right) e^{im\phi}$$

degree $(l-m)$

oscillates $(l-m)$ times in θ, m times in ϕ.

$P_{l0}(\cos\theta)$, $m=0$

$\sin^l\theta \cdot e^{\pm il\phi}$ $\quad m = \pm l$

$$\boxed{\int d\Omega \left|Y_{lm}\right|^2 = 1}$$

Strange and glorious thing "Addition Theorem"

$$\cos\gamma = \hat{r}\cdot\hat{r}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

$$P_\ell(\cos\gamma) = \sum_{\ell' m'} A_{\ell' m'}\, Y_{\ell' m'}(\theta,\phi) \qquad \left(\begin{array}{c}\text{as any fn}\\ \text{of angles}\end{array}\right)$$

choose: $\hat{r}' = \hat{z}$ $\quad (\theta' = 0)$.

$$\nabla^2 P_\ell(\cos\gamma) = \nabla^2\left(P_\ell(\cos\theta)\right)$$

$$= \left(\cancel{\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \cancel{\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}\right)$$

$$= \frac{1}{r^2}\left(-\ell(\ell+1)P_\ell\right) = -\frac{\ell(\ell+1)}{r^2}P_\ell(\cos\theta)$$

$\hat{r}\cdot\hat{r}'$ geometric, doesn't depend on axis.

$$\rightarrow \boxed{\nabla^2 P_\ell(\cos\gamma) = -\frac{\ell(\ell+1)}{r^2}P_\ell(\cos\gamma)}$$

$$P_\ell = \text{series} = \sum A_{\ell' m'} Y_{\ell' m'}$$

$$\nabla^2 P_\ell(\cos\gamma) = \sum_{\ell' m'} A_{\ell' m'}\left[-\frac{\ell'(\ell'+1)}{r^2}Y_{\ell' m'}\right]$$

$$\Rightarrow \sum_{\ell' m'} A_{\ell' m'}\,\ell'(\ell'+1)\,Y_{\ell' m'} = \sum_{\ell' m'} \ell(\ell+1) A_{\ell' m'} Y_{\ell' m'}$$

$$[\ell'(\ell'+1) - \ell(\ell+1)]A_{\ell' m'} = 0 \qquad \ell' \neq \ell \rightarrow A_{\ell' m'} = 0$$

only $\ell' = \ell$

$$P_\ell(\cos\gamma) = \sum_{m'=-\ell}^{\ell} A_{m'} Y_{\ell m'}(\theta,\phi)$$

Pick off A_m $\int d\Omega\, Y^*_{\ell m}(\theta,\phi)\, P_\ell(\cos\gamma)$

$$= \int d\Omega\, Y^*_{\ell m} \sum_{m'} A_{m'} Y_{\ell m'} = A_m$$

$$\boxed{A_m(\underbrace{\theta',\phi'}_{(know)}) = \int d\Omega\, Y^*_{\ell m}(\theta,\phi)\; P_\ell(\cos\gamma)}$$

recast: $\hat{r}'$ = axis, γ = angle from axis, ψ = azimuthal angle about axis.

$$d\Omega_{\theta,\phi} \Leftrightarrow d\Omega_{\gamma,\psi} = (\sin\gamma\, d\gamma\, d\psi)$$

$$Y_{\ell,0}(\gamma,\psi) = \sqrt{\frac{4\pi}{2\ell+1}}\, P_\ell(\cos\gamma)$$

$$A_m = \int d\Omega_{\gamma\psi}\, Y^*_{\ell m}(\theta(\gamma,\psi),\phi(\gamma,\psi))\, Y_{\ell,0}(\gamma,\psi)\cdot\sqrt{\frac{4\pi}{2\ell+1}}$$

$$Y_{\ell m}(\theta,\phi) = Y_{\ell m}(\theta(\gamma,\psi),\phi(\gamma,\psi)) = \sum_{\tilde{m}=-\ell}^{\ell} C_{\tilde{m}}\, Y_{\ell\tilde{m}}(\gamma,\psi)$$

$\to C_0$ term

at $\gamma=0$ $\theta'=\theta$, $\phi'=\phi$

$$Y_{\ell m}(\theta',\phi') = \sum_{\tilde{m}=-\ell}^{\ell} C_{\tilde{m}}\, Y_{\ell\tilde{m}}(\gamma=0) = C_0\, Y_{\ell,0}(\gamma=0)$$

$(\sin^{\tilde{m}}\gamma)$ $\tilde{m}=0$

$$Y_{lm}(\theta'', \phi'') = C_0\, Y_{l,0}(\gamma=0) = C_0 \cdot \sqrt{\frac{4\pi}{2l+1}}\, P_l(\cos\gamma = 1)$$

$$Y_{lm}(\theta, \phi) = C_{\tilde{m}}\, Y_{l\tilde{m}}(\gamma, \psi)$$

$$\int d\Omega\, \underline{\underline{Y^*_{l,0}(\gamma,\psi)}}\, Y_{lm}(\theta(\gamma,\psi), \phi(\gamma,\psi)) = C_0 = \sqrt{\frac{4\pi}{2l+1}}\, Y_{lm}(\theta', \phi')$$

$$\Rightarrow \int d\Omega \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\gamma)\, Y_{lm}(\theta, \phi) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{lm}(\theta', \phi')$$

$$\Rightarrow A_m = \int d\Omega\, Y^*_{lm}(\theta, \phi)\, P_l(\cos\gamma) = \frac{4\pi}{2l+1}\, Y^*_{lm}(\theta', \phi')$$

$$\boxed{P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y^*_{lm}(\theta', \phi')\, Y_{lm}(\theta, \phi)}$$

Addition theorem.

$$\underline{\underline{Y^*_{lm} = (-1)^m Y_{l,-m}}} \qquad (-m)+(+m) \to X + X^* \to \text{real.}$$

Either factor can be Y^*.

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$\ell=0$ $\quad P_0 = 1 = \left(\frac{4\pi}{2\ell+1}\right)\left(\frac{1}{\sqrt{4\pi}}\right)\left(\frac{1}{\sqrt{4\pi}}\right)$ ✓

$\ell=1$

$$\frac{4\pi}{3}\Bigg[\left(+\sqrt{\frac{3}{8\pi}}\sin\theta' e^{+i\phi'}\right)\left(+\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\right)$$

$$+\left(\sqrt{\frac{3}{4\pi}}\cos\theta'\right)\left(\sqrt{\frac{3}{4\pi}}\cos\theta\right)$$

$$+\left(-\sqrt{\frac{3}{8\pi}}\sin\theta' e^{-i\phi'}\right)\left(-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\right)\Bigg]$$

$$= \frac{1}{2}\sin\theta\sin\theta' e^{-i(\phi-\phi')}$$

$$+\cos\theta\cos\theta'$$

$$+\frac{1}{2}\sin\theta\sin\theta' e^{+i(\phi-\phi')}$$

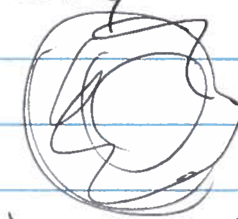
$$= \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') = \cos\gamma = P_1$$ ✓

$\theta'=\theta,\ \phi'=\phi$ $\quad \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell}\left|Y_{\ell m}(\theta,\phi)\right|^2 \cancel{\frac{2\ell+1}{4\pi}} = P_\ell(1) = 1.$

$$\boxed{\frac{1}{2\ell+1}\sum_{m=-\ell}^{\ell}\left|Y_{\ell m}\right|^2 = \frac{1}{4\pi}}$$

Green's Function: between two spheres

$V = \{a < r < b\}$



$$\nabla'^2 G = -4\pi\delta^{(3)}(x - x') = 0 \qquad A = \varepsilon$$

$$G_< = \sum_{lm} \left(A^<_{lm} r'^l + \frac{B^<_{lm}}{r'^{l+1}}\right) Y_{lm}(\theta', \phi') \qquad r' < r$$

$$G_> = \sum_{l,m} \left(A^>_{lm} r'^l + \frac{B^>_{lm}}{r'^{l+1}}\right) Y_{lm}(\theta', \phi') \qquad r' > r$$

$r' = a$: $\quad A^<_{lm} a^l + B^<_{lm} \frac{1}{r^{l+1}} = 0 \rightarrow A^<_{lm}\left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right)$

$r' = b$: $\quad A^>_{lm} a^l + B^>_{lm} \frac{1}{r'^{l+1}} = 0 \rightarrow B^>_{lm}\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right)$

$r' = r$: $\quad A^<_{lm}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right) = B^>_{lm}\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)$

$$G = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} \left(r_<^l - \frac{a^{2l+1}}{r_>^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) Y_{lm}(\theta', \phi')$$