10/5/2015

$$\nabla^2 G = -4\pi\,\delta^{(3)}(\vec{x}-\vec{x}')$$

$$a < r < b$$

$$G = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) Y_{lm}(\theta',\phi')$$

$$\delta^{(3)}(\vec{x}-\vec{x}') = \frac{\delta(r-r')}{r'^2}\,\underline{\delta(\cos\theta-\cos\theta')}\;\delta(\phi-\phi')$$

$$= \frac{\delta(r-r')}{r^2}\left[\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \underline{Y_{lm}^*(\theta,\phi)\,Y_{lm}(\theta,\phi')}\right]$$

$$\left(P_l(\cos\gamma) = \underline{\frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y_{lm}^*(\theta',\phi')\,Y_{lm}(\theta,\phi)}\right)$$

$\otimes$

②

$$\nabla'^2 G = \frac{1}{r'^2}\frac{\partial}{\partial r'}\left(r'^2 \frac{\partial G}{\partial r'}\right) + \frac{1}{r'^2}\frac{1}{\sin\theta'}\frac{\partial}{\partial \theta'}\left(\sin\theta' \frac{\partial G}{\partial \theta'}\right)$$

$$+ \frac{1}{r'^2}\frac{1}{\sin^2\theta'}\frac{\partial^2 G}{\partial \phi'^2} = -4\pi \frac{\delta(r-r')}{r'^2}\, \delta(\cos\theta - \cos\theta')\, \delta(\phi-\phi')$$

Integrate : $r' = \underline{r-\varepsilon}$ to $r' = \underline{r+\varepsilon}$

$$\int_{r-\varepsilon}^{r+\varepsilon} dr' \left[\frac{\partial}{\partial r'}\left(r'^2 \frac{\partial G}{\partial r'}\right) + (\underbrace{\theta'}_{\to\,\varepsilon\to 0}) + (\underbrace{\phi'}_{\to\,\varepsilon\to 0}) \right]$$

$$= r'^2 \left(\frac{\partial G}{\partial r'}\right)_{r'=r-\varepsilon}^{r'=r+\varepsilon} = -4\pi\, \delta^{(2)}(\Omega - \Omega')$$

$$\sum_{l,m} r'^2 \frac{\partial}{\partial r'}\left[C_{lm}\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)\right] Y_{lm}(\theta',\phi')$$

$$= -4\pi \sum_{l,m} Y^*_{lm}(\theta,\phi)\, Y_{lm}(\theta',\phi')$$

Term-by-term.

$$C_{lm}\left[\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)\frac{\partial}{\partial r'}\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right)\right.$$

$$\left. - \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)\frac{\partial}{\partial r'}\left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right)\right]$$

$$= -4\pi\, Y^*_{lm}(\theta,\phi)$$

③

$$C_{lm} r^2 \left[\left(r^l - \frac{a^{2l+1}}{r^{l+1}} \right) \left(-\frac{l+1}{r^{l+2}} - \frac{l r^{l-1}}{b^{2l+1}} \right) - \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) \left(l r^{l-1} + \frac{(l+1) a^{2l+1}}{r^{l+2}} \right) \right] = -4\pi Y^*_{lm}$$

$$C_{lm} r^2 \left[-\frac{(l+1)}{r^2} - \cancel{\frac{l r^{2l-1}}{b^{2l+1}}} + \cancel{\frac{(l+1) a^{2l+1}}{r^{2l+3}}} + \frac{l}{r^2} \frac{a^{2l+1}}{b^{2l+1}} - \left(\frac{l}{r^2} + \cancel{\frac{(l+1) b^{2l+1}}{r^{2l+3}}} - \cancel{\frac{l r^{2l-1}}{b^{2l+1}}} - \frac{(l+1)}{r^2} \frac{a^{2l+1}}{b^{2l+1}} \right) \right]$$

$$= C_{lm} \left(-(l+1) - l + \frac{l \cdot a^{2l+1}}{b^{2l+1}} + \frac{(l+1) a^{2l+1}}{b^{2l+1}} \right)$$

$$= -(2l+1) C_{lm} \left(1 - \frac{a^{2l+1}}{b^{2l+1}} \right) = -4\pi Y^*_{lm}$$

$$C_{lm} = \frac{4\pi}{2l+1} \frac{1}{1 - \left(\frac{a}{b}\right)^{2l+1}} Y^*_{lm}$$

$$G = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^{l+1}}{b^{2l+1}} \right)}{\left(1 - \left(\frac{a}{b}\right)^{2l+1} \right)} \times Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)$$

3.125

Interesting things:

a → 0, b → ∞

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi)$$

$$= \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \quad \text{as before}$$

$$= \frac{1}{|\vec{x}-\vec{x}'|}$$

b → ∞, a fixed

$$\left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}}\right)$$

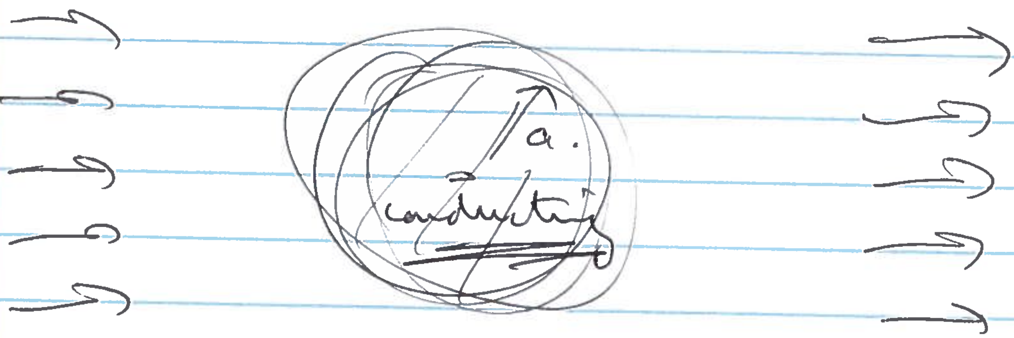
$$= \frac{r_<^l}{r_>^{l+1}} - \frac{a^{2l+1}}{r_<^{l+1} r_>^{l+1}} = \frac{r_<^l}{r_>^{l+1}} - \frac{a^{2l+1}}{r'^{l+1} r^{l+1}}$$

$$= \frac{r_<^l}{r_>^{l+1}} + \frac{(-a/r')(a^2/r')^l}{r^{l+1}}$$

↑ image $q' = -a/r' \, q$ @ $a^2/r' < r$

exterior of sphere $(r, r' > a)$

b fixed, a → 0 interior of sphere $r, r' < b$.



$\vec{E}_0 = E_0 \hat{z}$

$\Phi_0 = -E_0 z$

$= -E_0 r \cos\theta$

$$\Phi = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \qquad m=0$$

$r \to \infty$. $\boxed{A_1 = -E_0}$ $\quad A_l = 0 \quad l \geq 2$

$r \to a$ $\quad \cancel{A_0 = 0}$ $\quad A_l a^l + \dfrac{B_l}{a^{l+1}} = 0 \qquad l \geq 1.$

$l \geq 2 \quad A_l = B_l = 0.$

$$A_1 \cdot a + \frac{B_1}{a^2} = 0 \qquad B_1 = -A_1 a^3 = +E_0 a^3.$$

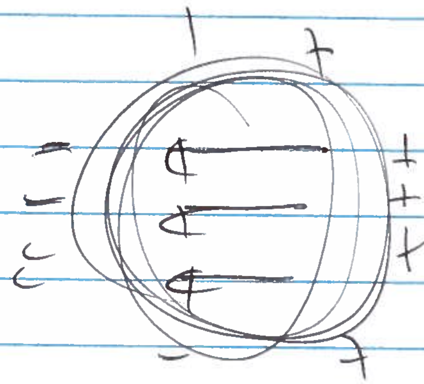
$l = 0 \qquad A_0 + \dfrac{B_0}{r} \qquad B_0 \to Q \to 0 \qquad A_0 = \text{constant} \to 0$

$$\boxed{\Phi = -E_0 \left(r - \frac{a^3}{r^2} \right) \cos\theta}$$

$\Phi(a)=0$. equipotential. (grounded)

$$E_r = -\frac{\partial\Phi}{\partial r} = \left(E_0 + \frac{2E_0 a^3}{r^3}\right)\cos\theta$$

$$\Delta E_r\Big|_{r=a} = +3E_0\cos\theta = \sigma/\varepsilon_0 \qquad \boxed{\sigma = 3\varepsilon_0 E_0\cos\theta}$$



$\sigma_0\cos\theta$

Superposition: shell by itself gives $\boxed{\vec{E}_{in} = -\vec{E}_0}$

$$\vec{E}_{in} = -\frac{\sigma_0}{3\varepsilon_0}\hat{z}$$