10/7/2015.

Cylindrical Coordinates. $x = \rho\cos\phi,\ y = \rho\sin\phi,\ z$

$$\vec{\nabla} = \hat{\rho}\frac{\partial}{\partial\rho} + \frac{\hat{\phi}}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}.$$

$$\vec{\nabla}\cdot\left(A_\rho\hat{\rho} + A_\phi\hat{\phi} + A_z\hat{z}\right)$$

$$= \frac{\partial A_\rho}{\partial\rho}\hat{\rho}\cdot\hat{\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}\hat{\phi}\cdot\hat{\phi} + \frac{A_\rho}{\rho}\frac{\partial\hat{\rho}}{\partial\phi}\cdot\hat{\phi} + \frac{\partial A_z}{\partial z}\hat{z}\cdot\hat{z}$$

$$\boxed{\nabla^2\Phi = \frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}}$$

$$\Phi = R(\rho)\,Q(\phi)\,Z(z).$$

$$\frac{1}{\Phi}\nabla^2\Phi = \frac{1}{R}\left(R'' + \frac{1}{\rho}R'\right) + \frac{Q''}{Q} + \frac{Z''}{Z}\quad \begin{cases} -k^2 & e^{\pm ikz} \\ +k^2 & e^{\pm kz}\end{cases}$$

$-m^2$ $\quad Q = e^{\pm im\phi}$ $\quad$ **Both**

$Q = e^{im\phi}$ $\quad Z = e^{\pm kz}$

$$\frac{1}{R}\left(R'' + \frac{1}{\rho}R'\right) - \frac{m^2}{\rho^2} + k^2 = 0.$$

$$\frac{1}{k^2}R'' + \frac{1}{k\rho}\frac{1}{k}R' + \left(1-\frac{m^2}{k^2\rho^2}\right)R = 0$$

let $x = k\rho$

$m \Rightarrow \nu$

$$\boxed{\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1-\frac{\nu^2}{x^2}\right)R = 0}$$

Behaviors

$x \to 0$ $\quad R'' + \frac{1}{x}R' - \frac{\nu^2}{x^2}R = 0$

$R \sim x^\alpha$ $\quad \alpha(\cancel{\alpha-1}) + \cancel{\alpha} - \nu^2 = 0$ $\quad \alpha^2 = \nu^2$ $\quad \boxed{\alpha = \pm\nu}$

$\boxed{R \sim x^{\pm\nu}}$

$x \to \infty$ $\quad R'' + R = 0$. $\quad R \sim \sin x, \cos x.$

Let

$R \sim \sin(x+\delta)\,x^p$.

$R' = p x^{p-1}\sin + x^p \cos$

$R'' = p(p-1)x^{p-2}\sin + 2p x^{p-1}\cos - x^p \sin$

$$\cancel{-x^p\sin} + 2p x^{p-1}\cos + p(p-1)x^{p-2}\sin$$
$$+ p x^{p-2}\sin + x^{p-1}\cos + \left(\cancel{1} - \frac{\nu^2}{x^2}\right)x^p \sin = 0$$

$(2p+1)\,x^{p-1}\cos(x+\delta) + O(x^{p-2}) = 0$.

$p = -\frac{1}{2}$ $\quad R \sim \frac{1}{\sqrt{x}}\sin(x+\delta)$

③

Series: $R = x^{\alpha} \sum_{j=0}^{\infty} a_j x^j$

$R' = \sum (j+\alpha) a_j x^{j+\alpha-1}$

$R'' = \sum (j+\alpha)(j+\alpha-1) a_j x^{j+\alpha-2}$

$$\sum_{j=0}^{\infty} \Big[(j+\alpha)(j+\alpha-1) a_j x^{j+\alpha-2} + \frac{1}{x}(j+\alpha) a_j x^{j+\alpha-1} + a_j x^{j+\alpha} - \alpha^2 a_j x^{j+\alpha-2} \Big] = 0.$$

Let $j+\alpha = j'+\alpha-2$ $\boxed{j' = j+2}$

$j=0$ $j'=-2$

$j=1$ $j'=-1$

$$(\alpha^2 - \nu^2) a_0 x^{\alpha-2} + ((\alpha+1)^2 - \nu^2) a_1 x^{\alpha-1}$$

$$+ \sum_{j'=0}^{\infty} \left[((j+\alpha)^2 - \nu^2) a_j + a_{j-2} \right] x^{j+\alpha-2} = 0.$$

$(\alpha^2 - \nu^2) a_0 = 0$, $[(\alpha+1)^2 - \nu^2] a_1 = 0.$

$a_0 \neq 0 \rightarrow \alpha^2 = \nu^2$ $\alpha = \pm\nu$. $a_1 = 0$

$a_1 \neq 0. \rightarrow (\alpha+1)^2 = \nu^2$ $\alpha = -1 \pm \nu$ $a_0 = 0$ $\leftarrow$ Same series.

$a_{j-2} = -[(j+\alpha)^2 - \nu^2] a_j$

$= -(j^2 + 2\alpha j + \cancel{\alpha^2 - \nu^2}) a_j$

$$\boxed{a_j = -\frac{a_{j-2}}{j^2+2\alpha j}} \qquad j = 2k.$$

$$a_{2k} = -\frac{a_{(2k-2)}}{4k^2+4\alpha k} = -\frac{1}{4}\,\frac{a_{2(k-1)}}{k(k+\alpha)}$$

$$\frac{1}{k!} = \frac{1}{k}\,\frac{1}{(k-1)!} \qquad \frac{1}{(k+\alpha)!} = \frac{1}{(k+\alpha)}\,\frac{1}{(k+\alpha-1)!}$$

$$a_{2k} = \frac{(-1)^k}{2^{2k}}\,\frac{1}{k!}\,\frac{\alpha!}{(\alpha+k)!}\cdot a_0.$$

Bernoulli

<u>convention</u>: $a_0 = \left[2^\alpha \cdot \alpha!\right]^{-1}$

$\alpha = \pm\nu$

$$\boxed{\begin{aligned} J_\nu(x) &= \left(\frac{x}{2}\right)^{\nu} \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!}\,\frac{1}{(k+\nu)!}\left(\frac{x}{2}\right)^{2k} && (3.82) \\ J_{-\nu}(x) &= \left(\frac{x}{2}\right)^{-\nu} \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!}\,\frac{1}{(k-\nu)!}\left(\frac{x}{2}\right)^{2k} && (3.83) \end{aligned}}$$

converge (rapidly) for any x. $\frac{(x/2)^2}{k(k+\alpha)} \to 0$ $(k \geq x/2)$.

$\nu \neq$ integer J_ν, $J_{-\nu}$ <u>independent</u>

(5)

Problem: if $\nu = n$ = integer:

$$J_{-n}(x) = \left(\frac{x}{2}\right)^{-n} \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \frac{1}{(k-n)!} \left(\frac{x}{2}\right)^{2k}$$

$\frac{1}{(-n)!} \to 0.$ $\quad x! = x(x-1)!$

$$(\varepsilon!) = \varepsilon(\varepsilon-1)(\varepsilon-2)\ldots(\varepsilon-(n-1))(\varepsilon-n)!$$

$$\frac{1}{(\varepsilon-n)!} = \frac{1}{(\varepsilon!)}\left(\varepsilon(\varepsilon-1)\cdots(\varepsilon-(n-1))\right)$$

$$\simeq \varepsilon \frac{(-1)^{n-1}(n-1)!}{(0)!} \to 0.$$

$x!$ has single pole at $x=-n$ residue $= (-1)^n (n-1)!$

$$\cancel{J_{-n}(x)} \; J_{-n} = \left(\frac{x}{2}\right)^{-n} \sum_{k=n}^{\infty} (-1)^k \frac{1}{k!} \frac{1}{(k-n)!} \left(\frac{x}{2}\right)^{2k}$$

$k' = k-n$
$k = k'+n$

$$= \left(\frac{x}{2}\right)^{-n} \sum_{k'=0}^{\infty} (-1)^{k'+n} \frac{1}{(k'+n)!} \frac{1}{(k')!} \left(\frac{x}{2}\right)^{2k'+2n}$$

$$= (-1)^n \left(\frac{x}{2}\right)^{2n} \left(\frac{x}{2}\right)^{-n} \sum_{k'=0}^{\infty} (-1)^{k'} \frac{1}{(k')!} \frac{1}{(k'+n)!} \left(\frac{x}{2}\right)^{2k'}$$

$$\boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

ν near $-n$ $\qquad J_{-\nu}$ near $J_{-n} = (-1)^n J_n + O(\epsilon)$.

$\nu = -n + \epsilon$

$$(-1)^\nu J_\nu - J_{-\nu} = O(\epsilon)$$

$$\frac{(-1)^\nu J_\nu - J_{-\nu}}{\epsilon} = \text{finite}$$

$$\boxed{N_\nu(x) \equiv \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}} \qquad (3.85).$$

JDJ: $N_\nu(x)$ — "Bessel function of 2nd kind"
elsewhere $Y_\nu(x)$. — "Neumann Function" 1867

H. Weber 1873.

Heinrich Martin Weber. mathematician

(Heinrich Friedrich Weber. physicist,
First advisor of A.E.
(50 years out of date, no Maxwell).