10/9/2015 $(x \to 0)$

$|x| \ll |\nu|$

$$J_\nu \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu = \frac{1}{m!}\left(\frac{x}{2}\right)^m$$

$$N_\nu \to -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu = -\frac{(m-1)!}{\pi}\left(\frac{2}{x}\right)^m$$

$$N_0 \to \frac{2}{\pi}\left(\ln\frac{x}{2} + \gamma_E\right) \qquad \gamma_E = 0.5772156649\ldots$$

$(x \to \infty)$

$$J_\nu \to \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_\nu \to \sqrt{\frac{2}{\pi x}} \cdot \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$H_\nu^{(1,2)} = J_\nu \pm i N_\nu \to \sqrt{\frac{2}{\pi x}}\, e^{\pm i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)}$$

Hankel

$$N_\nu \simeq -\frac{2^\nu\,\Gamma(\nu)}{\pi x^\nu} - \frac{2^{\nu-2}}{x^{\nu-2}}\,\frac{\Gamma(\nu)}{\nu-1}$$

$$N_2 \simeq -\frac{4}{\pi x^2} - \frac{1}{\pi} + \frac{\left(\log\frac{x}{2} + \gamma_E - \frac{3}{4}\right)}{4\pi} x^2 + \ldots$$

②

Bessel $\frac{d^2 J_\nu}{dx^2} + \frac{1}{x}\frac{dJ_\nu}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) J_\nu = 0$

Legendre $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dP}{d\theta}\right) + \left(\ell(\ell+1) - \frac{m^2}{\sin^2\theta}\right) P = 0$

θ small, ℓ big

$\sin\theta \simeq \theta \quad + O(\theta^2)$ relative

$\ell(\ell+1) \simeq (\ell+\tfrac{1}{2})^2 \quad + O\left(\tfrac{1}{\ell^2}\right)$

$$\frac{1}{\theta}\frac{d}{d\theta}\left(\theta \frac{dP}{d\theta}\right) + \left[(\ell+\tfrac{1}{2})^2 - \frac{m^2}{\theta^2}\right] P = 0$$

$$P_\ell^m(\cos\theta) \simeq J_m\left((\ell+\tfrac{1}{2})\theta\right)$$

$m=0$

$$\boxed{P_\ell(\cos\theta) \simeq J_0\left((\ell+\tfrac{1}{2})\theta\right)}$$

$P_{100}(0) = \frac{(99)!!}{(100)!!} = \frac{1.261\times10^{28}}{1.584\times10^{29}} = 0.0795$

Recursions

$F = J, N, H^{(1)}, \ldots$

$$F_{\nu-1} + F_{\nu+1} = \frac{2\nu}{x} F_\nu$$

near $x=0$ unstable (do in reverse)

$$F_{\nu-1} - F_{\nu+1} = 2\frac{dF_\nu}{dx}$$

$\nu=0$

$J_{-1} + J_{+1} = 0 \qquad \boxed{J_{-1} = -J_1}$

$J_{-1} - J_{+1} = -2J_1 = 2\frac{dJ_0}{dx} \qquad \boxed{J_0' = -J_1}$

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Special cases. $\nu = \frac{1}{2}$

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$N_{\frac{1}{2}}(x) = -\sqrt{\frac{2}{\pi x}} \cos x .$$

$x \to 0$

$$J_{\frac{1}{2}} \to \frac{1}{(\frac{1}{2})!} \left(\frac{x}{2}\right)^{1/2} = \sqrt{\frac{2x}{\pi}}$$

$\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$

$$N_{\frac{1}{2}} \to -\frac{\Gamma(\frac{1}{2})}{\pi} \left(\frac{2}{x}\right)^{1/2} = -\sqrt{\frac{2}{\pi x}}$$

$x \to \infty$

$$J_{\frac{1}{2}} \to \sqrt{\frac{\pi}{2x}} \cos\left(x - \frac{1}{2}\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= \sqrt{\frac{\pi}{2x}} \cos\left(x - \frac{\pi}{2}\right)$$

$\cancel{\cos x \cos\frac{\pi}{2}} + \sin x \sin\frac{\pi}{2}$

$$N_{\frac{1}{2}} \to \sqrt{\frac{\pi}{2x}} \sin\left(x - \frac{1}{2}\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$\cancel{\sin x \cos\frac{\pi}{2}} - \cos x \sin\frac{\pi}{2}$

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Oscillate $\rightarrow$ zeroes $x_{\nu n}$, x_{mnd}

$\nu = 0$ $\quad x_{01} = 2.4048255\,69\ldots$ (table).

new. $\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) = 0.$

$$x - \frac{\nu\pi}{2} - \frac{\pi}{4} = \left(n + \tfrac{1}{2}\right)\pi$$

$$x_{\nu n} \rightarrow \left(n + \tfrac{1}{2}\right)\pi + \frac{\nu\pi}{2} + \frac{\pi}{4}$$

$$= \left(n + \tfrac{3}{4}\right)\pi + \nu\frac{\pi}{2}$$

$$= \left(\frac{3\pi}{4} + \nu\frac{\pi}{2}\right) + n\pi$$

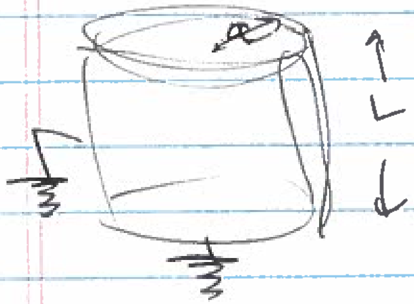
$\nu = 0$ $\quad \frac{3\pi}{4} = 2.356$ $\quad \frac{7\pi}{4} = 3.930$

$x_{01} = 2.405$ $\quad 3.892$

$\nu = 1$ $\quad \frac{5\pi}{4} = 3.927$

$x_{11} =$

$J_m(r)$ complete, orthonormal in many ways.



cylinder $0 < \rho < a$, $0 < z < L$.

$\Phi(\rho = a) = 0 \qquad \Phi(\rho = 0) = 0$

$$\Phi(z = L) = V_s(\rho, \phi)$$

$$\Phi = \sum_{k,m} A_{km} \underbrace{J_m(k\rho)}_{\text{regular}} e^{im\phi} \underbrace{\sinh kz}_{\text{vanishes @ } z=0}$$

$J_m(ka) = 0 \rightarrow ka = x_{mn} \qquad k_{mn} = \frac{x_{mn}}{a}$

$\left\{ J_m\left(\frac{x_{mn}\rho}{a}\right) \right\}$ orthogonal for different n

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dJ_m}{d\rho}\left(\frac{x_{mn}\rho}{a}\right)\right]+\left(\frac{x_{mn}^2}{a^2}-\frac{m^2}{\rho^2}\right)J_m=0$$

mply $J_{mn}\left(\frac{x_{mn}'\rho}{a}\right)$, integrate $\int_0^a \rho\,d\rho$.

$$\int_0^a \cancel{\rho}\,d\rho\; J_m\left(\frac{x'\rho}{a}\right)\left[\frac{1}{\cancel{\rho}}\frac{d}{d\rho}\left(\rho\frac{dJ}{d\rho}\left(\frac{x\rho}{a}\right)\right)+\left(\frac{x^2}{a^2}-\frac{m^2}{\rho^2}\right)J_m\right]=0$$

parts.

surface $=\left[J_m\left(\frac{x'\rho}{a}\right)\cdot\rho\,\frac{dJ_m}{d\rho}\left(\frac{x\rho}{a}\right)\right]_0^a$

$=J_m(x')\,\cancel{a}\left(\frac{x}{\cancel{a}}\right)J_m'(x)$

$=\boxed{x\,J_m(x')\,J_m'(x)}$ $x_{mn}'\to 0$

$$\int_0^a \rho\,d\rho\left[-\frac{dJ_m}{d\rho}\left(\frac{x'\rho}{a}\right)\cdot\rho\,\frac{dJ_m}{d\rho}\left(\frac{x\rho}{a}\right)\right]+\left(\frac{x^2}{a^2}-\frac{m^2}{\rho^2}\right)J_mJ_n\Big]$$

$x_{mn}\leftrightarrow x_{mn}'$, subtract

$$\left(\frac{x_{mn}^2}{a^2}-\frac{x_{mn}'^2}{a^2}\right)\int_0^a\rho\,d\rho\;J_m\left(\frac{x\rho}{a}\right)J_m\left(\frac{x'\rho}{a}\right)=0$$

$x_n\neq x_n'\rightarrow$ orthogonal