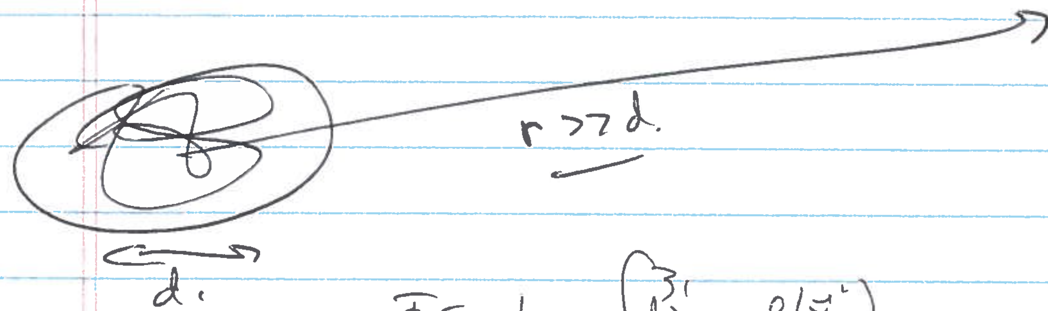
10/19/2015 Chapter 4. Stuff → solid state physics.



$$\Phi = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{(r^2+r'^2-2\vec{x}\cdot\vec{x}')^{1/2}} = \frac{1}{r}\,\frac{1}{\left(1+\frac{r'^2}{r^2}-\frac{2\vec{x}\cdot\vec{x}'}{r^2}\right)^{1/2}}$$

$$\cong \frac{1}{r}\left(1+\left(-\tfrac{1}{2}\right)\left(\frac{r'^2}{r^2}-\frac{2\vec{x}\cdot\vec{x}'}{r^2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{2\vec{x}\cdot\vec{x}'}{r^2}\right)^2+\cdots\right)$$

$$= \frac{1}{r}+\frac{\vec{x}\cdot\vec{x}'}{r^3}+\frac{1}{2}\,\frac{3(\vec{x}\cdot\vec{x}')^2-r^2r'^2}{r^5}+\cdots$$

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \rho(\vec{x}')\left[\quad\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int d^3x' \rho(\vec{x}') + \frac{x_i}{r^3}\int d^3x' \, x'_i \, \rho(\vec{x}') + \frac{1}{2}\frac{x_i x_j}{r^5}\int d^3x' \rho(\vec{x}')\left(3x'_i x'_j - \delta_{ij} r'^2\right)\right]$$

$$\Phi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2} Q_{ij}\frac{x_i x_j}{r^5} + \cdots\right)$$

$$Q = \int d^3x'\, \rho(\vec{x}')$$ charge

$$\vec{p} = \int d^3x'\, \vec{x}'\rho(\vec{x}')$$ dipole moment (center of mass)

$$Q_{ij} = \int d^3x' \left(3x'_i x'_j - \delta_{ij} r'^2\right) \rho(\vec{x}')$$ quadrupole moment

(moment of inertia)

Q = scalar (1 number)

$\vec{p}$ = vector (3 numbers)

Q_{ij} = matrix (tensor) $3\times 3 = 9$.

$Q_{ij} = Q_{ji}$ (symmetric) $\to -3 \to 6$

$Q_{ii} = \delta_{ij} Q_{ij} = \mathrm{Tr}\, Q_{ij} = 0 \to -1$

$\to 5$

Q_{ijk}. $3\times3\times3 = 27$

symmetric. $\frac{3\cdot4\cdot5}{3!} = 10$

traceless. $Q_{ijj} = 0$. (-3)

$\to 7$

$\left(\begin{matrix} Q_{111}, Q_{222}, Q_{333} \\ Q_{112}, Q_{113}, Q_{211}, Q_{233}, Q_{311}, Q_{322} \\ Q_{123} \end{matrix} \right)$

$Q_{111} + Q_{221} + Q_{331} = 0$

$Q_{112} + Q_{222} + Q_{332} = 0$

$Q_{113} + Q_{223} + Q_{333} = 0$

③

$Q^{(3)}_{ij\ldots k}\,\hat{r}_i \ldots \hat{r}_k \qquad \nabla^2 \to -\frac{l(l+1)}{r^2}. \qquad \text{rank} = l.$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\phi') Y_{lm}(\theta,\phi)$$

$$\Phi = \frac{1}{4\pi\varepsilon_0} \sum_{l,m} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}} \underbrace{\int d^3x'\, r'^l\, Y^*_{lm}(\theta',\phi')\,\rho(\vec{x}')}_{q_{lm}}$$

Not different.

$$q_{00} = \int d^3x'\, \rho(\vec{x}') \cdot 1 \cdot \frac{1}{\sqrt{4\pi}} = \frac{1}{\sqrt{4\pi}} Q$$

$$q_{1,\pm1} = \int d^3x'\, \rho(\vec{x}') \left(\mp\sqrt{\frac{3}{8\pi}}\right) r \sin\theta\, e^{\pm i\phi} = \mp\sqrt{\frac{3}{8\pi}}\,(p_x \pm i p_y)$$

$$q_{1,0} = \int d^3x'\, \rho(\vec{x}') \cdot r \cdot \cos\theta \sqrt{\frac{3}{4\pi}} = \frac{\sqrt{3}}{\sqrt{4\pi}}\, p_z$$

Q_{ij} = linear combination of q_{2m}

$Q_{ijk} \Longleftrightarrow q_{3m}$

$\Phi_Q = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$ know from before.

$$\boxed{\Phi_{\vec p} = \frac{1}{4\pi\epsilon_0}\frac{\vec p\cdot\vec x}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}}$$

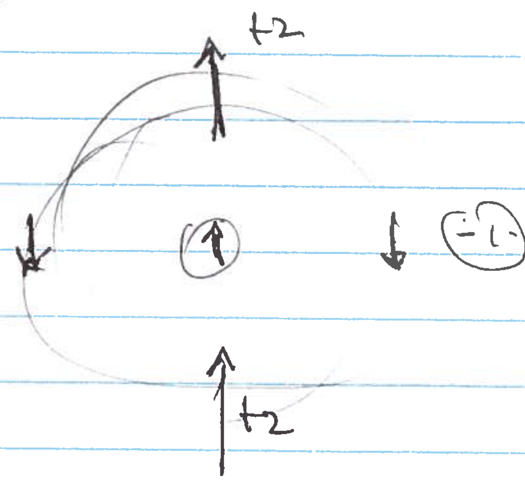
2nd in importance to Q.

$$\vec E_{\vec p} = -\vec\nabla\Phi_{\vec p} = -\nabla_i\left(\frac{1}{4\pi\epsilon_0}\frac{p_j x_j}{r^3}\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\left(\frac{p_j\delta_{ij}}{r^3} - 3p_j x_j\left(\frac{x_i}{r^3}\right)\right)$$

$$\boxed{\vec E_{\vec p} = \frac{1}{4\pi\epsilon_0}\frac{3\hat r(\hat r\cdot\vec p) - \vec p}{r^3}} + \delta\text{-function}$$

$$E_r = -\frac{\partial\Phi}{\partial r} = +\frac{2}{4\pi\epsilon_0}\frac{p\cos\theta}{r^3}$$



$$E_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = +\frac{1}{4\pi\epsilon_0}\frac{p\sin\theta}{r^3}$$

$$\vec E = \frac{1}{4\pi\epsilon_0}\frac{p}{r^3}\left(2\hat r\cos\theta + \hat\theta\sin\theta\right)$$

Everything has a "multiple expansion"

$$W = \int d^3x\ \rho(\vec{x})\ \Phi(\vec{x})$$ local ρ in distant Φ.

$$= \int d^3x\ \rho(\vec{x}) \left[\Phi(0) + (\vec{\nabla}_i \Phi)_0 \cdot x^i + \frac{1}{2} (\nabla_i \nabla_j \Phi)_0 \cdot x_i x_j + \cdots \right)$$

$$= \Phi_0 \cdot \int d^3x\ \rho(\vec{x}) + (\nabla_i \Phi)_0 \cdot \int d^3x\ \rho(\vec{x}) \cdot x_i$$

$$+ \frac{1}{2} (\nabla_i \nabla_j \Phi)_0 \cdot \int d^3x\ x_i x_j \cdot \rho(\vec{x})$$

$$\rightsquigarrow \frac{1}{3} \int d^3x \left(3x_i x_j - \delta_{ij} \right) x^2 \rho \ + \ \frac{1}{3} \delta_{ij} \int d^3x\ x^2 \rho$$

$$(\nabla^2 \Phi)_0 = 0.$$

$$W = Q \cdot \Phi_0 - \vec{p} \cdot \vec{E}_0 + \frac{1}{6} (\nabla_i \nabla_j \Phi)_0 \cdot Q_{ij} + \cdots$$

Force, torque $\rightarrow$ HW.

Useful exercise. $\langle \vec{E} \rangle_R$

$$\langle \vec{E}(\vec{x}) \rangle_R = \frac{1}{\frac{4\pi}{3}R^3} \int_0^R d^3x \, \vec{E}(\vec{x}) = \frac{1}{V} \int_0^R d^3x \left(-\vec{\nabla}\Phi\right)$$

Aside. $\int_V d^3x \, \vec{\nabla} \cdot \vec{A} = \oint_S d^2a \, \hat{n} \cdot \vec{A}$

let $\vec{A} = \vec{a}\,\Phi$, $\vec{a}$ = some constant vector.

$$\int_V d^3x \, \vec{\nabla} \cdot \vec{A} = \int_V d^3x \, \vec{\nabla} \cdot (\vec{a}\,\Phi) = \left(\int_V d^3x \, \vec{\nabla}\Phi \right) \cdot \vec{a}$$

$$\oint_S d^2a \, \hat{n} \cdot \vec{A} = \left(\oint_S d^2a \, \hat{n}\,\Phi \right) \cdot \vec{a}$$

for any $\vec{a}$ $\rightarrow$ $\boxed{\int_V d^3x \, \vec{\nabla}\Phi = \oint_S d^2a \, \hat{n}\,\Phi}$

2nd cover of Jackson. line 2