10/28/15.

Chapter 5 "Magnetostatics"

magnetic field $\vec{B}$ { "magnetic flux density" Wb m^{-2} / "magnetic induction" / magnetic field

$[B]$ = Tesla = T = Wb m^{-2}

Earth field known in Shang dynasty (O(3k) yr ago)
(foundations aligned with magnetic north)

Compass (from china) (c. 1190 a.d.)

$$\boxed{\vec{N} = \vec{\mu} \times \vec{B}} \qquad \vec{F} = q\,\vec{v} \times \vec{B}$$

Ørsted (Copenhagen). 4/21/1820?
Ampère (Paris). 9/11/1820 — 11/6/1820

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \int I\, d\vec{l}' \times \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}}$$

(wire)

$\mu_0 = 4\pi \times 10^{-7}$ T A^{-1} m^{-1} NA^{-2} H m^{-1}

~10% in 150 y
50 km/year migration

Equator 0.31 gauss
N pole 0.58 gauss
S pole 0.6? gauss

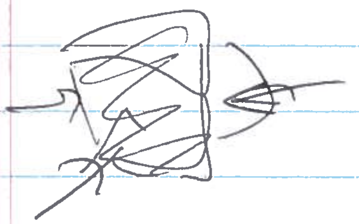
dipole

Volume. $I\, d\vec{l} \to \vec{J}\, d^3x$.

$\vec{J}$ = current density (per cross section area)

$[J] = C\,m^{-2}s^{-1}$ $I = \int \vec{J}\cdot\hat{n}\, d^2a$

magnetostatics. "stationary flow" (out = in)



$$\frac{dQ}{dt} = -\oint_S d^2a\, \hat{n}\cdot\vec{J} = -\int d^3x\, \vec{\nabla}\cdot\vec{J}$$

$$= \int d^3x \left(\frac{\partial\rho}{\partial t}\right)$$

$$\boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0}$$ Equation of continuity (conservation of charge)

stationary → $\frac{dQ}{dt} = 0$ → $\vec{\nabla}\cdot\vec{J} = 0$ (statics only)

$$\boxed{\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x'\, \vec{J}(\vec{x}') \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}$$

Biot-Savart (5.14) (should be in Sec. 5.2)

$\frac{1}{4\pi\epsilon_0} \to \frac{\mu_0}{4\pi}$ $\rho \to \vec{J}$ $\times$

Long straight wire $\qquad I\,d\vec{l}\,' = I\hat{z}\,dz'$

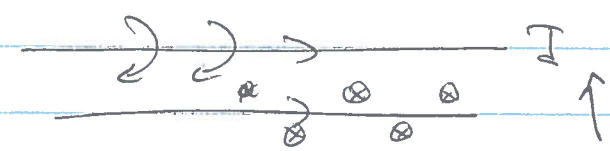
$$\vec{B}(\rho,\phi,z) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} I\hat{z}\,dz' \times \frac{\left(\rho\hat{\rho} + \cancel{z\hat{z}} - \cancel{\rho'\hat{\rho}'} - \cancel{z'\hat{z}}\right)}{\left[\rho^2 + \cancel{\rho'^2} - \cancel{2\rho\rho'\cos(\phi-\phi')} + (z-z')^2\right]^{3/2}}$$

$$= \frac{\mu_0}{4\pi}\, I\,(\hat{z}\times\hat{\rho})\,\rho \int_{-\infty}^{\infty}\frac{dz'}{\left[\rho^2+(z-z')^2\right]^{3/2}} \to \frac{2}{\rho^2}$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}} \qquad \left(\text{cf. } \vec{E} = \frac{\lambda}{2\pi\epsilon_0}\hat{\rho}\right)$$

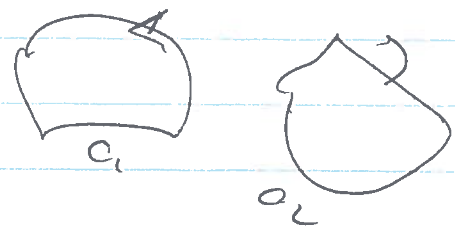
$$d\vec{F} = dq\,\vec{v}\times\vec{B} = (I\,dt)\,\vec{v}\times\vec{B} = I(\vec{v}\,dt\times\vec{B})$$

$$= I\,d\vec{l}\times\vec{B} \qquad \boxed{\vec{F} = \int d^3x\,\vec{J}\times\vec{B}} \quad \left(\text{cf. } \int d^3x\,\rho\vec{E}\right)$$



Parallel currents attract

Closed loops.



$$\vec{F}_{12} = \underline{\vec{F}_{\substack{\text{on } 1 \\ \text{from } 2}}} = \oint_{C_1} I_1 \vec{dl}_1 \times \vec{B}_2$$

$$= \oint_{C_1} I_1 \vec{dl}_1 \times \frac{\mu_0}{4\pi} \oint_{C_2} I_2 \, dl_2 \times \frac{(\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|^3}$$

$$= \frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{\vec{dl}_1 \times (\vec{dl}_2 \times \vec{x}_{12})}{|\vec{x}_{12}|^3}$$

$$\vec{dl}_1 \times (\vec{dl}_2 \times \vec{x}_{12}) = \vec{dl}_2 \, (\vec{dl}_1 \cdot \vec{x}_{12}) - \vec{x}_{12} (\vec{dl}_1 \cdot \vec{dl}_2)$$

$$\vec{dl}_1 \cdot \left(\frac{\vec{x}_{12}}{|x_{12}|^3}\right) = -\vec{dl}_1 \cdot \vec{\nabla}_1 \left(\frac{1}{|\vec{x}_{12}|}\right) = -d\left(\frac{1}{|x_{12}|}\right)$$

$$\oint_{C_1} \vec{dl}_1 \cdot \vec{\nabla}_1 \frac{1}{|x_{12}|} = \oint_{C_1} d\left(\frac{1}{x_{12}}\right) = 0$$

$$\boxed{\vec{F}_{12} = -\frac{\mu_0 I_1 I_2}{4\pi} \oint_{C_1} \oint_{C_2} (\vec{dl}_1 \cdot \vec{dl}_2) \frac{\vec{x}_{12}}{|x_{12}|^3}}$$

Symmetric, except in sign $\vec{F}_{12} = -\vec{F}_{21}$
(equal, opposite reaction)

$\boxed{\text{Parallel Currents Attract}}$

curl, div $\longrightarrow$ under integral

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = A_i\left(\varepsilon_{ijk} B_j C_k\right) = \varepsilon_{ijk} A_i B_j C_k$$
$$= \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{\nabla}\cdot\left[\vec{J}(\vec{x}')\times\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right] = -\vec{J}\cdot\left(\vec{\nabla}\times\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)\right) = 0$$

$$\vec{\nabla}\times\vec{\nabla}\frac{1}{|x|} \Longrightarrow \vec{\nabla}\times(\vec{\nabla}f) = \varepsilon_{ijk}\nabla_j\nabla_k f = 0$$

$d^2 = 0$

$\vec{\nabla}\cdot\vec{B} = 0$ No magnetic monopoles

5.17

≠ Price, Cabrera

$$\vec{\nabla}\times\left(\vec{J}(\vec{x}')\times\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\right) = \vec{J}(\vec{x}')\Big(\underbrace{\vec{\nabla}\cdot\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)}_{4\pi\,\delta^{(3)}(\vec{x}-\vec{x}')}\Big) - (\vec{J}\cdot\vec{\nabla})\left(\frac{\vec{x}_{12}}{x_{12}^3}\right)$$

surface $\qquad \nabla\cdot J = 0$

$$\rightarrow\ -\int d^3x'\ \vec{J}(\vec{x}')\cdot\vec{\nabla}'\left(\frac{\vec{x}}{x^3}\right) = \int d^3x'\left[\nabla_j'\left(J_j\frac{x_i'}{x^3}\right) - (\nabla_j' J_j)\frac{x_i'}{x^3}\right]$$

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int d^3x'\ \vec{J}(\vec{x}')\ 4\pi\,\delta^{(3)}(\vec{x}-\vec{x}')$$

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ Ampère's law

5.22

room for $\frac{\partial}{\partial t}$

$\vec{\nabla}\cdot\vec{B} = 0 \quad \longleftrightarrow \quad \boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ (5.27)

Biot-Savart. $\vec{B} = \frac{\mu_0}{4\pi}\int d^3x' \, \vec{J}\times\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \frac{\mu_0}{4\pi}\int d^3x' \, \vec{J}\times\left(-\vec{\nabla}\right)\frac{1}{|\vec{x}-\vec{x}'|}$

$$= \vec{\nabla}\times\left(\frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}} \quad (5.32)$$

$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$

can always add $\vec{\nabla}\Lambda$ to $\vec{A}$ (can add const. to Φ).

$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda \qquad \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \cancelto{0}{\vec{\nabla}\times\vec{\nabla}\Lambda} = \vec{\nabla}\times\vec{A}$

"Gauge Freedom". $\vec{A}$ has 3 d.f.

$\vec{\nabla}\times\vec{A}$ uses only two.

$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\Lambda \quad \Rightarrow$ can choose Λ $\quad \vec{\nabla}\cdot\vec{A}' = 0$

$(\nabla^2\Lambda = -\vec{\nabla}\cdot\vec{A})$

choose $\vec{\nabla}\cdot\vec{A} = 0$ (statics). $\qquad \boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$ (5.31)

$\vec{\nabla}\cdot\int d^3x' \, \vec{J}\times\frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \int d^3x' \, \vec{J}\cdot\left(\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}\right)$ $= 0$.

$= -\int d^3x' \, \vec{J}\cdot\left(\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|}\right) = \int d^3x' \left[-\vec{\nabla}'\cdot\left(\frac{\vec{J}}{|\vec{x}-\vec{x}'|}\right) + \frac{1}{|\vec{x}-\vec{x}'|}(\vec{\nabla}'\cdot\vec{J})\right.$