10/30/2015

$$\boxed{\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x'\ \vec{J}(\vec{x}') \times \left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)}$$ (Biot - Savart) (5.14)

$$\vec{F} = q\vec{v} \times \vec{B} \rightarrow \int d^3x\ \rho\vec{v} \times \vec{B} = \int d^3x\ \vec{J} \times \vec{B}$$

$$\vec{J} = \rho\vec{v}$$

$\vec{\nabla}\cdot\vec{\nabla}\times$

$\vec{\nabla}\cdot(\vec{\nabla}\times$

$\vec{\nabla}\times\vec{\nabla}$

$\vec{\nabla}\cdot\vec{\nabla}\times$

$$\vec{\nabla}\cdot\vec{B} = 0 \quad (5.17)$$

$$\vec{\nabla}\times\left(\vec{\nabla}\frac{1}{|x|}\right) = 0 \qquad d^2 = 0$$

$$\vec{\nabla}\times\vec{B} \rightarrow \vec{\nabla}\times\left(\vec{J}(\vec{x}') \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)$$

$$= \vec{J}(\vec{x}')\left(\vec{\nabla}\cdot\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)\right) - \left(\vec{J}(\vec{x}')\cdot\vec{\nabla}\right)\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)$$

$$\int d^3x'\ \vec{J}(\vec{x}')\cdot\vec{\nabla}\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right) = -\int d^3x'\ \vec{J}(\vec{x}')\cdot\vec{\nabla}'\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)$$

$$= -\int d^3x' \left\{ \underbrace{\vec{\nabla}'\cdot\left[\vec{J}\left(\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\right)\right]}_{\oint \rightarrow 0} - \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\underbrace{\left(\vec{\nabla}'\cdot\vec{J}(\vec{x}')\right)}_{0\ \text{(statics)}} \right\}$$

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int d^3x'\ \vec{J}(\vec{x}')\ 4\pi\,\delta^{(3)}(\vec{x}-\vec{x}')$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$$ Ampère's Law (statics) (5.22)

$\longrightarrow$ I

$\frac{\partial}{\partial\phi} \to 0$ $\quad$ $\frac{\partial}{\partial z} \to 0$ $\quad$ $B_\rho \to 0$ (flux) $\quad$ $B_z \to 0$ $\times$.

$$\oint_C \vec{B}\cdot d\vec{l} = 2\pi\rho B_\phi = \mu_0 \int da\, \hat{n}\cdot\vec{J} = \mu_0 I$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \longleftrightarrow \vec{B} = \vec{\nabla}\times\vec{A} \qquad (5.27)$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = \varepsilon_{ijk}\nabla_i\nabla_j A_k \qquad d^2 = 0$$

Biot savart $\vec{B} = \frac{\mu_0}{4\pi}\int d^3x'\, \vec{J}(\vec{x}')\times\left(-\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}\right)$

$$= \vec{\nabla}\times\left(\frac{\mu_0}{4\pi}\int d^3x'\, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right)$$

$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int d^3x'\, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}}$$

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$$

$\uparrow$ inconvenient.

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"<u>Gauge freedom</u>". $\vec{A}$ has 3 d.f.
$\vec{\nabla}\times\vec{A}$ uses only two.

can always add $\vec{\nabla}\Lambda$ to $\vec{A}$. (can add C to Φ)

$$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(\vec{A}+\vec{\nabla}\Lambda) = \vec{\nabla}\times\vec{A}.$$

$$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\Lambda.$$

<u>choose</u> Λ such that $\vec{\nabla}\cdot A' = 0$. $\quad \nabla^2\Lambda = -\vec{\nabla}\cdot\vec{A}$
know how to solve.

<u>Choose</u> $\vec{\nabla}\cdot\vec{A} = 0$ $\rightarrow$ $\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$ (5.31).

$$\vec{\nabla}\cdot\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} = \int d^3x' \, \vec{J}(\vec{x}')\cdot\vec{\nabla}\frac{1}{|x|}$$

$$= -\int d^3x' \, \vec{J}(\vec{x}')\cdot\vec{\nabla}'\frac{1}{|x-\vec{x}'|}$$

$$= -\int d^3x' \, \vec{\nabla}'\cdot\left(\frac{\vec{J}}{|\vec{x}|}\right) + \int d^3x' \frac{1}{|\vec{x}|}\,\cancel{\vec{\nabla}'\cdot\vec{J}(x')}$$

$\cancel{\oint} \to 0 \qquad \to 0$

④

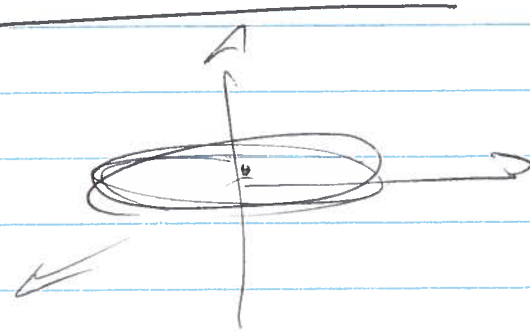
Straight wire.

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{d^3x'\,\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} = \frac{\mu_0}{4\pi}\int \frac{I\hat{z}\,dz'}{|\vec{x}-\vec{x}'|}$$

$$= \frac{\mu_0}{4\pi} I\hat{z}\int \frac{dz'}{(\rho^2+(z-z')^2)^{1/2}} = -\frac{\mu_0 I}{2\pi}\hat{z}\ln\rho \quad + \text{constant}$$

$$\vec{\nabla}\times\vec{A} = \left(-\frac{\mu_0 I}{2\pi}\right)\hat{z}\times\left(\frac{\hat{\rho}}{\rho}\right) = \frac{\mu_0 I}{2\pi\rho}\hat{\phi}$$

Current loop



$$\vec{J} = I\hat{\phi}\cdot\delta(\rho-a)\cdot\delta(z)$$

$$z = r\cos\theta$$

$$\rho = r\sin\theta$$

$$\delta(z) = \frac{1}{r}\delta(\cos\theta) = \frac{1}{r\sin\theta}\delta(\theta-\tfrac{\pi}{2})$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{r'^2\,dr'\,\sin\theta'\,d\theta'\,d\phi'\;\left(\frac{I\hat{\phi}'}{a}\right)\delta(r'-a)\,\delta(\theta'-\frac{\pi}{2})}{(r^2+r'^2-2rr'\cos\gamma)^{1/2}}$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{Ia\,\hat{\phi}'\,d\phi'}{(r^2+a^2-2ar\sin\theta\cos(\phi-\phi'))^{1/2}}}$$

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$$\frac{1}{(r^2+a^2-2ar\sin\theta\cos(\phi-\phi'))^{1/2}} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)$$

$$= \frac{1}{r} + \frac{\vec{x}\cdot\vec{x}'}{r^3} + \frac{r_<^2}{r_>^3}P_2 + \cdots$$

$$\cancel{\frac{(\hat{x}\cdot r\sin\theta)\,a\cos(\phi-\phi')}{r^3}}$$

$$\boxed{\cos\gamma = \sin\theta\cos(\phi-\phi')}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int I a\,\hat{\phi}'\,d\phi'\left(\cancel{\frac{1}{r}} + \frac{a}{r^2}\sin\theta\cos(\phi-\phi') + \cdots\right)$$

$$= \frac{\mu_0}{4\pi}\frac{Ia^2}{r^2}\sin\theta\int_0^{2\pi}d\phi'\,\hat{\phi}'\cos(\phi-\phi')$$

$$\int d\phi'\,(-\hat{x}\sin\phi' + \hat{y}\cos\phi')\,(\cos\phi\cos\phi' + \sin\phi\sin\phi')$$

$$= -\hat{x}\cdot\sin\phi\,(\pi) + \hat{y}\cdot\cos\phi\cdot(\tfrac{1}{2}\cdot 2\pi)$$

$$\boxed{\vec{A} \to \frac{\mu_0}{4\pi}\,I\pi a^2\cdot\frac{\sin\theta}{r^2}\,\hat{\phi}}$$

$$\vec{\nabla}\times\vec{A} = \frac{\hat{r}}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\phi\right) + \hat{\theta}\left[-\frac{1}{r}\frac{\partial}{\partial r}\right](rA_\phi)$$

$$= \frac{\hat{r}}{r\cancel{\sin\theta}}\left(2\cancel{\sin\theta}\cos\theta\,\frac{\mu_0}{4\pi}\frac{I\pi a^2}{r^2}\right)$$

$$+\ \hat{\theta}\left(-\frac{1}{r}\right)\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\right)}\left(-\frac{1}{r^2}\right)\frac{\mu_0}{4\pi}I\pi a^2\,\sin^2\theta$$

$$\vec{B}_{(r\to\infty)} = \frac{\mu_0}{4\pi}\,I\pi a^2\,\frac{\left(2\hat{r}\cos\theta + \hat{\theta}\sin\theta\right)}{r^3}$$

cf. $\mathcal{O}\left(\frac{\cos\theta}{r^2}\right)$, Dipole. $m = I\pi a^2$

In full

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{Ia\,\hat{\phi}'\,d\phi'}{(r^2+a^2-2ar\sin\theta\cos(\phi-\phi'))^{1/2}}$$

$\phi'-\phi = \phi''$

$\phi' = \phi''+\phi$

$$= \frac{\mu_0}{4\pi}\int\frac{Ia\,d\phi''\left[-\hat{x}\sin(\phi''+\phi) + \hat{y}\cos(\phi''+\phi)\right]}{(r^2+a^2-2ar\sin\theta\cos\phi'')^{1/2}}$$

$$\int d\phi''\sin\phi''\,F(\cos\phi'') = \int_{-\pi}^{0}\ominus + \int_{0}^{\pi}\oplus = 0.$$

$$\sin(\phi''+\phi) = \cancel{\sin\phi''\cos\phi} + \underline{\cos\phi''}\sin\phi$$

$$\cos(\phi''+\phi) = \underline{\cos\phi''}\cos\phi - \cancel{\sin\phi''\sin\phi}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I a\, d\phi'' \left(-\hat{x}\cos\phi''\sin\phi + \hat{y}\cos\phi''\cos\phi\right)}{\left(r^2 + a^2 - 2ar\sin\theta\cos\phi''\right)^{1/2}}$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi} I a \hat{\phi} \int \frac{d\phi'' \cdot \cos\phi''}{\left(r^2 + a^2 - 2ar\sin\theta\cos\phi''\right)^{1/2}}}$$

<u>Elliptic Integrals</u> $K(m)$, $E(m)$