11/2/2015

$\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{A} = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$

$\vec{B}_0 = B_0 \hat{z}$

$\boxed{\vec{A} = \frac{1}{2} \vec{B}_0 \times \vec{x}}$

$\vec{\nabla} \times \vec{A} = \frac{1}{2} \vec{\nabla} \times (\vec{B}_0 \times \vec{x})$

$= \frac{1}{2}\left(\vec{B}_0 (\vec{\nabla} \cdot \vec{x}) - (\vec{B}_0 \cdot \vec{\nabla})\vec{x}\right)$

$= \frac{1}{2} \vec{B}_0 (3 - 1) = \vec{B}_0$

$\vec{A} = \frac{1}{2} B_0 \hat{z} \times (x\hat{x} + y\hat{y} + \cancel{z\hat{z}})$

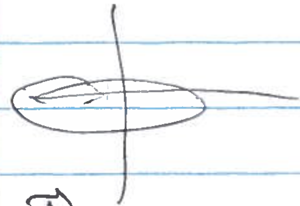
$\vec{A} = \frac{1}{2} B_0 (x\hat{y} - y\hat{x})$ $= \frac{1}{2} B_0 (r\sin\theta\cos\phi\, \hat{y} - r\sin\theta\sin\phi\, \hat{x})$

$\vec{A} = \frac{1}{2} B_0 r \sin\theta\, \hat{\phi}$ $\quad l = 1$

$\vec{A} = B_0 x \hat{y}$ $\quad \vec{A} = -B_0 y \hat{x}$

$\frac{1}{2}(x\hat{y} - y\hat{x}) - (-y\hat{x}) = \frac{1}{2}(x\hat{y} + y\hat{x}) = \frac{1}{2} \nabla (xy)$

②



$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I a \hat{\phi}' \, d\phi'}{\left(r^2 + a^2 - 2ar \sin\theta \cos(\phi - \phi')\right)^{1/2}}$$

$$= \frac{\mu_0}{4\pi} \int \frac{I a \left(-\hat{x} \sin\phi' + \hat{y} \cos\phi'\right) d\phi'}{\left(\quad \right)^{1/2}} \qquad \phi'' = \phi - \phi'$$

$$= \frac{\mu_0}{4\pi} \int \frac{I a \left(-\hat{x} \cdot (\cancel{\sin\phi'' \cos\phi} + \cos\phi'' \sin\phi) + \hat{y} (\cos\phi'' \cos\phi + \cancel{\sin\phi'' \sin\phi}) \right) d\phi''}{\left(r^2 + a^2 - 2ar \sin\theta \cdot \cos\phi''\right)^{1/2}}$$

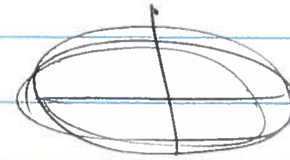
$$\int d\phi'' \sin\phi'' \cdot F(\cos\phi'') = \int_{-\pi}^{0} \ominus + \int_{0}^{\pi} \oplus = 0.$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{I a \left(-\hat{x} \sin\phi \cos\phi'' + \hat{y} \cos\phi \cos\phi''\right) d\phi'}{\left(\quad - \quad - \quad - \quad \right)^{1/2}}$$

$$\vec{A} = \frac{\mu_0}{4\pi} I a \hat{\phi} \int \frac{d\phi'' \cos\phi''}{\left(r^2 + a^2 - 2ar \cos\phi''\right)^{1/2}}.$$

B

Circumference of Ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad x = a\cos\theta \qquad (a > b).$$
$$y = a\sin\theta$$

$$dl^2 = dx^2 + dy^2 = a^2 \cdot \sin^2\theta \, d\theta^2 + b^2 \cdot \cos^2\theta \, d\theta^2$$

$$= \left[a^2 - (a^2 - b^2)\cos^2\theta\right] d\theta^2$$

$$C = 4 \cdot a \cdot \int_0^{\pi/2} d\theta \sqrt{1 - \left(\frac{a^2 - b^2}{a^2}\right)\cos^2\theta}$$

$$\boxed{C = 4a \cdot E\left(\frac{a^2 - b^2}{a^2}\right)}$$

E = elliptic integral "second kind".

$b = a$, circle. $C = 2\pi a$ $\quad E(0) = \frac{\pi}{2}$

$b = 0$ degenerate. (line segment) $C = 4a$ $\quad E(1) = 1$

$\theta \to \frac{\pi}{2} - \theta$ (complementary) $\quad \cos\theta \to \sin\theta$

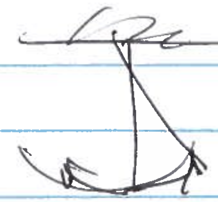
$$\boxed{E(m) = \int_0^{\pi/2} d\theta \sqrt{(1 - m\sin^2\theta)} = \int_0^1 dt \left(\frac{1 - mt^2}{1 - t^2}\right)^{1/2}}$$

mathematica, Aitken. $E(m)$

Jackson, Wolfram. $E(k)$, $k^2 = m$.

$$E(m) = \frac{\pi}{2} \, {}_2F_1\left(-\tfrac{1}{2}, \tfrac{1}{2}, 1; m\right)$$

Period of pendulum


max $= \theta_0$.

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1-\cos\theta) = mgl(1-\cos\theta_0)$$

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0) = \frac{4g}{l}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)$$

$\sin^2\frac{\theta}{2} = \frac{1}{2}(1-\cos\theta)$ $\qquad$ $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

$$T = 4\int_0^{\theta_0} dt = 4\int_0^{\theta_0}\frac{d\theta}{\dot{\theta}} = 4\sqrt{\frac{l}{4g}}\int_0^{\theta_0}\frac{d\theta}{\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)^{1/2}}$$

Let $\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\cdot\sin\varphi$ (astronomy trick!)

$$\cos\frac{\theta}{2}\cdot\frac{1}{2}d\theta = \sin\frac{\theta_0}{2}\cdot\cos\varphi\, d\varphi.$$

$$T = 2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{2\cdot\cancel{\sin\frac{\theta_0}{2}}\cdot\cos\varphi\, d\varphi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\varphi}}\cdot\cancel{\frac{1}{\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta_0}{2}\sin^2\varphi\right)^{1/2}}}$$

$\cos\frac{\theta_0}{2}$

$$\boxed{T = 4\sqrt{\frac{l}{g}}\,K(m)} \qquad m = \sin^2\frac{\theta_0}{2}$$

$m \to 0$ $\quad K \to \frac{\pi}{2}$ $\qquad\qquad$ $m \to 1.$ $\quad K \to \infty.$

$$K(m) = \int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-m\cdot\sin^2\theta}} = \int_0^1\frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}$$

8

$$\boxed{\frac{dE}{dm} = \frac{E(m) - K(m)}{2m}} \qquad \boxed{\frac{dK}{dm} = \frac{E - (1-m)K}{2m(1-m)}}$$

$$K(m) \simeq \frac{\pi}{2} + \frac{\pi}{8}\cdot m + \frac{9\pi}{128} m^2 + \dots$$

$$E(m) = \frac{\pi}{2} - \frac{\pi}{8} m - \frac{3\pi}{128} m^2 + \dots \qquad (m \to 0)$$

$$K(m) = -\frac{1}{2}\log(1-m) + \log 4 + O(1-m) \qquad (m \to 1)$$

$$E(m) = 1. \quad (\text{non differentiable})$$

$$\cos^2\frac{\phi''}{2} = \frac{1}{2}(1+\cos\phi'')$$

$$\hat{A} = \frac{\mu_0}{4\pi} I a \hat{\phi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi'' \frac{\cos\phi''}{(r^2+a^2-2ar\sin\theta\cos\phi'')^{1/2}}$$

$$= \frac{\mu_0}{4\pi} I a \hat{\phi} \int_{-\pi}^{\pi} \frac{2\cdot d\left(\frac{\phi''}{2}\right)\cdot\left(2\cos^2\frac{\phi''}{2} - 1\right)}{\left[r^2+a^2-2ar\sin\theta\left(2\cos^2\frac{\phi''}{2}-1\right)\right]^{1/2}}$$

$$= \frac{\mu_0 I a \hat{\phi}}{4\pi} \frac{1}{(r^2+a^2+2ar\sin\theta)^{1/2}}$$

$$\times 2 \int_{-\pi/2}^{\pi/2} \frac{d\phi'\cdot(2\cos^2\phi' - 1)}{\left(1 - \frac{4ar\sin\theta}{r^2+a^2+2ar\sin\theta}\cos^2\phi'\right)^{1/2}}$$

$$\hat{A} = \frac{4\mu_0 I a \hat{\phi}}{(r^2+a^2+2ar\sin\theta)^{1/2}} \int_0^{\pi/2} \frac{d\psi' \left[\left(\frac{-2}{m}\right)(1-m\cos^2\psi') + \left(\frac{2}{m}-1\right)\right]}{(1-m\cos^2\psi')}$$

$m \geq 0$

$$\boxed{m = \frac{4ar\sin\theta}{r^2+a^2+2ar\sin\theta}} \qquad 1-m = \frac{r^2+a^2-2ar\sin\theta}{r^2+a^2+2ar\sin\theta} \geq 0$$

$$\boxed{\hat{A} = \frac{\mu_0}{4\pi} I a \hat{\phi} \frac{4}{(r^2+a^2+2ar\sin\theta)^{1/2}} \left[\frac{(2-m)K(m) - 2E(m)}{m}\right]}$$

$r \gg a$ — far away

$a \gg r$ — near center

$\sin\theta \ll 1$ — near axis

$m \to 0$.

$O(\frac{1}{m}), O(1)$ cancel

$$\frac{(2-m)K-2E}{m} \longrightarrow \frac{\pi m}{16} + \frac{3\pi}{64}m^2 + \cdots$$

$$\hat{A} \to \frac{\mu_0}{4\pi} I a^2 \hat{\phi} \cdot \frac{\cancel{4}}{(r^2+a^2+2ar\sin\theta)^{1/2}} \cdot \frac{\pi}{\cancel{16}} \frac{\cancel{4}ar\sin\theta}{(r^2+a^2+2ar\sin\theta)}$$

$$\boxed{\hat{A} \to \frac{\mu_0}{4\pi} I \pi a^2 \hat{\phi} \frac{r\sin\theta}{(r^2+a^2+2ar\sin\theta)^{3/2}}}$$