11/4/2015 $\hat{A} = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$

$$\longrightarrow \frac{\mu_0}{4\pi} I a \hat{\phi} \cdot \frac{4}{(r^2+a^2+2ar\sin\theta)^{1/2}} \left[\frac{(2-m)K(m) - 2E(m)}{m} \right]$$

$$m = \frac{4ra}{r^2+a^2+2ar\sin\theta}$$

JDJ k^2 $m \geq 0$

$$1-m = \frac{r^2+a^2-2ar\sin\theta}{r^2+a^2+2ar\sin\theta} \geq 0.$$

$r \gg a$. (far away)

$a \gg r$ (near center)

$\sin\theta \ll 1$. (near axis)

m small.

$\frac{1}{m}$?

$$\frac{(2-m)K - 2E}{m} \to \frac{\pi m}{16} + \frac{3\pi}{64} m^2 + \ldots$$

$$\vec{A} \to \frac{\mu_0}{4\pi} I\pi a^2 \hat{\phi} \frac{r\sin\theta}{(r^2+a^2+2ar\sin\theta)^{3/2}}$$

Far: $\rightarrow \frac{\mu_0}{4\pi} I\pi a^2 \cdot \frac{\sin\theta\,\hat{\phi}}{r^2}$ → dipole as before

Center: $\rightarrow \frac{\mu_0}{4\pi} I\pi a^2 \cdot \frac{r\sin\theta\,\hat{\phi}}{a^2} = \frac{\mu_0 I}{4a}\, r\sin\theta\,\hat{\phi}$

$$\vec{B} = \frac{\mu_0 I}{4a}\left(\hat{r}\,\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta \cdot r) - \frac{\hat{\theta}}{r}\frac{\partial}{\partial r}(r^2\sin\theta)\right)$$

$$= \frac{\mu_0 I}{4a}\left(2\cos\theta\,\hat{r} - 2\sin\theta\,\hat{\theta}\right) = \underline{\frac{\mu_0 I}{2a}\hat{z}}$$

$$\frac{\mu_0}{4\pi}\int Ia\,\hat{\phi}'\,d\phi' \times \frac{(-a\hat{\rho}')}{(a^2)^{3/2}} = \frac{\mu_0}{4\pi}\frac{Ia\,2\pi}{a^2}\hat{z} \checkmark$$

Near ring: $r = a + \Delta r$

$\theta = \frac{\pi}{2} + \Delta\theta$

$$m = \frac{4a(a+\Delta r)(1-\frac{1}{2}(\Delta\theta)^2)}{(a+\Delta r)^2 + a^2 - 2a(a+\Delta r)(1-\frac{1}{2}\Delta\theta^2)} \simeq 1 - \frac{\Delta r^2 + a^2\Delta\theta^2}{4a^2}$$

$$\vec{A} \approx \frac{\mu_0}{4\pi} Ia\,\hat{\phi} \cdot \frac{4}{(4a^2)^{\frac{1}{2}}} \frac{(2-m)K(m) - 2}{(1)}$$

$$= \frac{\mu_0}{2\pi} I\hat{\phi}\left(-\frac{1}{2}\log\frac{(\Delta d)^2}{4a^2}\right) = -\frac{\mu_0}{2\pi} I\hat{n}\log\frac{\Delta d}{2a}$$

Problems. | Biot-Savart — $\vec{A}(\vec{x})$ ↔ (coulombs, $\Phi(\vec{x})$)

Outside currents: $\left.\begin{array}{l}\nabla\times\vec{B}=0\\ \nabla\cdot\vec{B}=0\end{array}\right\}$ $\begin{array}{l}\vec{B}=-\nabla\Phi\\ \nabla^2\Phi=0\end{array}$

$$\Phi_m = \sum \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

$\Delta B_\perp = 0$ $\Delta B_\parallel = \mu_0 K$ (directions)

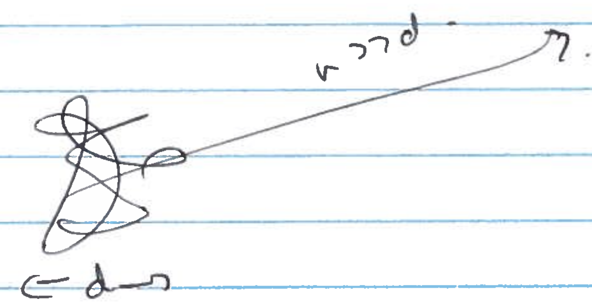
$\nabla\times\vec{B}$ $\hat{n}\times\Delta\vec{B} = \mu_0\vec{K}$

Green's Function

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}Y^*_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi)$$

Friday.

Localized sources



$$\vec{A} = \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} = \frac{\mu_0}{4\pi}\int d^3x' \, \vec{J}(\vec{x}')\left(\frac{1}{r} + \frac{\vec{x}\cdot\vec{x}'}{r^3} + \cdots\right).$$

Theorem:

$$\vec{\nabla}\cdot(fg\vec{J}) = \nabla_k(fgJ_k) = (\nabla_k f)gJ_k + f(\nabla_k g)J_k + fg\cancel{\nabla_k J_k}$$

$$\int d^3x' \, \vec{\nabla}'\cdot(fg\vec{J}) = \oint_{R'\to\infty} d^2a' \, \hat{n}\cdot(fg\vec{J}) = 0.$$

$$\Rightarrow \boxed{\int d^3x' \, (g\vec{\nabla}'f + f\vec{\nabla}'g)\cdot\vec{J}(\vec{x}') = 0}$$

(statics)

1st term: $f=1$ $\quad g = x'_i \quad \nabla'_k f = 0 \quad \nabla'_k g = \delta_{ik}$

$$\int d^3x' \, (0 + \delta_{ik})J_k = \int d^3x' \, J_i = 0$$

$$\boxed{\int d^3x' \, \vec{J}(\vec{x}') = 0}$$

→ on average, current doesn't go anywhere.

$\boxed{\text{No } \frac{1}{r} \text{ term}}$ No monopole term.

2nd term $\quad f = x'_i \quad g = x'_j \quad \nabla'_k f = \delta_{ik} \quad \nabla'_k g = \delta_{jk}$

$$\int d^3x' \left(x'_i \delta_{jk} + x'_j \delta_{ik} \right) J_k = 0$$

$$\boxed{\int d^3x' \left(x'_i J_j + x'_j J_i \right) = 0}$$

$$\frac{\mu_0}{4\pi} \int d^3x' \frac{(\vec{x}' \cdot \vec{x})}{r^3} \vec{J} = \frac{\mu_0 x_j}{4\pi r^3} \int d^3x' \, x'_j J_i$$

$$= \frac{\mu_0}{4\pi} \frac{x_j}{r^3} \int d^3x' \left[\cancel{\frac{1}{2}\left(x'_j J_i + x'_i J_j\right)} + \frac{1}{2}\left(x'_j J_i - x'_i J_j\right) \right]$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^3} \int d^3x' \, \frac{1}{2} \left((\vec{x} \cdot \vec{x}') \vec{J} - (\vec{x} \cdot \vec{J}) \vec{x}' \right)$$

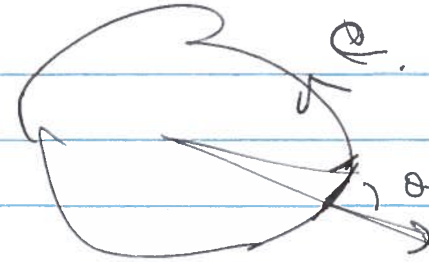
$$\vec{A} = \frac{\mu_0}{4\pi} \frac{1}{r^3} \int d^3x' \, \frac{1}{2} \left(\vec{x}' \times \vec{J} \right) \times \vec{x}$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{r^3}} \qquad \boxed{\vec{m} = \frac{1}{2} \int d^3x' \left(\vec{x}' \times \vec{J} \right)}$$

dipole

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} \vec{\nabla} \times \left(\vec{m} \times \frac{\vec{x}}{r^3} \right) = \frac{\mu_0}{4\pi} \left[\vec{m} \left(\vec{\nabla} \cdot \frac{\vec{x}}{r^3} \right) - \vec{\nabla} \left(\frac{\vec{m} \cdot \vec{x}}{r^3} \right) \right]$$

$$= \frac{\mu_0}{4\pi} \frac{3\hat{r}(\hat{r} \cdot \vec{m}) - \vec{m}}{r^3} + \frac{\mu_0}{4\pi} \left(4\pi - \frac{4\pi}{3} \right) \vec{m} \, \delta(\vec{x})$$



Plane Loop

$$\vec{J}\, d^3x \to I\, d\vec{l}$$

$$\vec{m} = I \oint_C \frac{1}{2}\, \underline{\vec{x} \times d\vec{l}}$$

↳ $(r)(dl)(\sin\theta)\,\hat{n}$

$$= I\hat{n} \oint \frac{1}{2} r \cdot dr_{\perp} = I\hat{n} \oint da = I\hat{n} \cdot A$$

$$\vec{m} = Ia \cdot \hat{n}$$

$$(\vec{p} = q\vec{d}).$$