11/23/2015

$$\vec\nabla\cdot\vec E = \rho/\varepsilon_0$$

$$\vec\nabla\times\vec E = -\frac{\partial\vec B}{\partial t} \qquad \leftarrow (-\vec J_m)$$

$$\vec\nabla\cdot\vec B = 0 \qquad \leftarrow (\rho_m)$$

$$\vec\nabla\times\vec B = \mu_0\vec J + \mu_0\varepsilon_0\frac{\partial\vec E}{\partial t}$$

$$\vec\nabla\cdot(\vec\nabla\times\vec B) = 0 = \mu_0\left(\frac{\partial\rho}{\partial t} + \vec\nabla\cdot\vec J\right)$$ continuity

$$\vec\nabla\cdot\vec B = 0 \rightarrow \boxed{\vec B = \vec\nabla\times\vec A}$$

Homogeneous

$$\vec\nabla\times\vec E + \frac{\partial\vec B}{\partial t} = \vec\nabla\times\left(\vec E + \frac{\partial\vec A}{\partial t}\right) = 0$$

$$\rightarrow \boxed{\vec E = -\vec\nabla\Phi - \frac{\partial\vec A}{\partial t}}$$

$$\vec\nabla\cdot\vec E = \vec\nabla\cdot\left(-\vec\nabla\Phi - \frac{\partial\vec A}{\partial t}\right) = \rho/\varepsilon_0.$$

$$\boxed{\nabla^2\Phi + \frac{\partial}{\partial t}(\vec\nabla\cdot\vec A) = -\rho/\varepsilon_0}$$

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

$$= \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial}{\partial t}\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right)$$

$$\boxed{\underbrace{\nabla^2\vec{A} - \mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}}_{} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A} + \mu_0\varepsilon_0\frac{\partial\vec{A}}{\partial t}\right) = -\mu_0\vec{J}}$$

Wave operator $= \Box^2 =$ D'Alembertian

traveling wave $F(x-ct)$.

$$\frac{\partial^2 F}{\partial x^2} = F''(x-ct) \qquad \frac{\partial^2 F}{\partial t^2} = (-c)^2 F''(x-ct).$$

$$\boxed{\frac{\partial^2 F}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = 0} \qquad \boxed{\mu_0\varepsilon_0 = \frac{1}{c^2}}$$

$$\mu_0\varepsilon_0 = \frac{\mu_0}{4\pi}\cdot(4\pi\varepsilon_0) = \frac{10^{-7}}{9\times10^9} = \frac{1}{(3\times10^8\,m/s)^2} = \frac{1}{c^2}$$

would be nice if $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t} = 0$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \qquad \Box^2\vec{A} = -\mu_0\vec{J}$$

$$\nabla^2\Phi + \frac{\partial}{\partial t}\left(-\frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right) = -\rho/\varepsilon_0 \qquad \Box^2\Phi = -\rho/\varepsilon_0.$$

Can we impose $\vec{\nabla}\cdot\vec{A} + \mu_0\epsilon_0 \frac{\partial \Phi}{\partial t} = 0$?

$\vec{B} = \vec{\nabla}\times\vec{A}$ $\qquad \boxed{\vec{A}' = \vec{A} + \vec{\nabla}\Lambda}$

$$\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \cancel{\vec{\nabla}\times\vec{\nabla}\Lambda} = \vec{B}$$

want: $\vec{E}' = -\vec{\nabla}\Phi' - \frac{\partial}{\partial t}\vec{A}'$

$$= -\vec{\nabla}\Phi' - \frac{\partial \vec{A}}{\partial t} - \frac{\partial}{\partial t}\vec{\nabla}\Lambda$$

$$= -\vec{\nabla}\left(\Phi' + \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

$\boxed{\Phi' = \Phi - \frac{\partial \Lambda}{\partial t}}$ $\qquad \Lambda = \Lambda(\vec{x}, t)$

"gauge" transformation

$$\vec{\nabla}'\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} = \vec{\nabla}\cdot\vec{A} + \nabla^2\Lambda + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2}$$

want $= 0$.

$$\Rightarrow \text{want } \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = -\left(\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right)$$

$$\boxed{\Box^2 \psi = -4\pi j}$$

Construct: $\Box^2 G = -4\pi\,\delta^{(4)}(\vec{x} - \vec{x}')\,\delta(t-t') = 0$ A.E.

$G = X(x)\,Y(y)\,Z(z)\,T(t)$.

$\frac{X''}{X} = -k_x^2 \qquad \frac{Y''}{Y} = -k_y^2 \qquad \frac{Z''}{Z} = -k_z^2 \qquad \frac{T''}{T} = -\omega^2,$

$G = \sum e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$$\left.\begin{aligned} \nabla^2 G &= -|\vec{k}|^2 G \\ \frac{\partial^2 G}{\partial t^2} &= -\omega^2 G \end{aligned}\right\} |\vec{k}|^2 = \frac{\omega^2}{c^2}$$

$$\cancel{G = \int \frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}\,\tilde{G}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}}$$

$$\Box^2 G = -4\pi\,\delta^{(4)} = -4\pi \int \frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}\, e^{i(\vec{k}\cdot\overset{d}{\Delta\vec{x}} - \omega \overset{d}{\Delta t})}$$

$$G = \int \frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}\,\tilde{G}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\Delta\vec{x} - \omega\cdot\Delta t)}$$

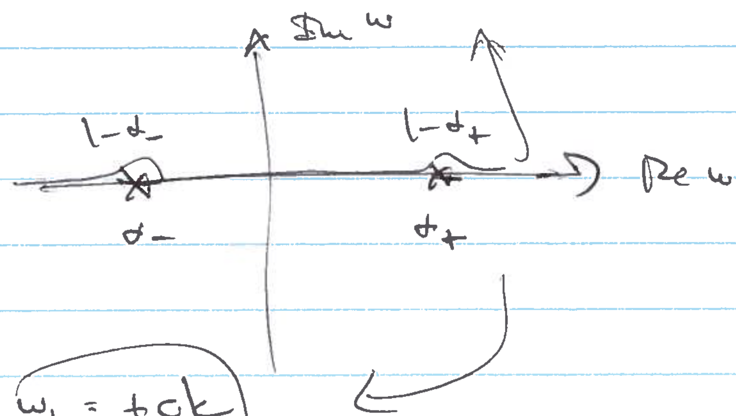
$$\Box^2 G = \int \frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}\left[-|\vec{k}|^2 + \frac{\omega^2}{c^2}\right]\tilde{G}(\vec{k},\omega)\, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\left(-|\vec{k}|^2 + \frac{\omega^2}{c^2}\right) G = -4\pi \qquad \text{"} \; G = \frac{4\pi}{\Box^2} \; \text{"}$$

$$G(\vec{x}, \vec{x}', t, t') = \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} \frac{-4\pi}{\frac{\omega^2}{c^2} - |\vec{k}|^2} e^{i(\vec{k}\cdot\Delta\vec{x} - \omega\Delta t)}$$

ω-integral first: $\displaystyle\int \frac{d\omega}{2\pi} \cdot \frac{-4\pi c^2}{\omega^2 - c^2k^2} e^{-i\omega\Delta t}$

complex contour integral



poles at $\omega_\pm = \pm ck$

$\Delta t > 0$. $e^{-i\omega\Delta t} = e^{-i(\mathrm{Re}\,\omega)\Delta t} e^{(\mathrm{Im}\,\omega)\Delta t}$

→ close contour in lower half plane

$e^{(\mathrm{Im}\,\omega)\Delta t} = e^{-|\mathrm{Im}\,\omega||\Delta t|} \to 0$

$$\int \oint \to (-)(2\pi i)(\delta_- R_- + \delta_+ R_+)$$

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$\Delta t < 0$, close contour in upper half-plane

$$e^{-i\omega t} = e^{-i(\mathrm{Re}\,\omega)\Delta t}\, e^{-|\mathrm{Im}\,\omega||\Delta t|} \to 0$$

$$\oint \to 2\pi i \cdot \left[(1-\alpha_-) R_- + (1-\alpha_+) R_+ \right]$$

$\alpha = \frac{1}{2} \to$ Cauchy principal value.

$\Delta t > 0$

$$\oint \frac{d\omega}{2\pi} \frac{-4\pi c^2}{(\omega-\omega_-)(\omega-\omega_+)} e^{-i\omega(\Delta t)}$$

R_- R_+

$$= -(2\pi i) \frac{(-4\pi c^2)}{(2\pi)} \left(\alpha_- \frac{e^{-i\omega_- t}}{2\omega_-} + \alpha_+ \frac{e^{i\omega_+ t}}{2\omega_+} \right)$$

$$= i \cdot 4\pi c^2 \cdot \left(\alpha_+ \frac{e^{-ick\Delta t}}{2ck} + \alpha_- \frac{e^{+ick\Delta t}}{-2ck} \right)$$

$$G(\Delta\vec{x}, \Delta t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{R}} \frac{4\pi i c^2}{2ck} \left(\alpha_+ e^{-ickt} - \alpha_- e^{+ickt} \right)$$

$$\int d\Omega\, e^{ikR\cos\theta} = \int d\phi \int_{-1}^{1} d\mu\, e^{ikR\mu}$$

$$= 2\pi \cdot \left(\frac{e^{ikR} - e^{-ikR}}{(ikR)} \right) = 4\pi \frac{\sin kR}{kR}$$