12/4/2015 Harmonic time dependence $e^{-i\omega t}$ $\mathrm{Re}(\cdot)$

(separation of variables, Fourier transform)

$\partial_t \to -i\omega$

$$\vec{E}(\vec{x},t) = \mathrm{Re}\{\vec{E}(\vec{x})\, e^{-i\omega t}\}$$
$$= \frac{1}{2}\left(\vec{E}(\vec{x})\, e^{-i\omega t} + \vec{E}^*(x)\, e^{+i\omega t}\right).$$

$\omega > 0$

Energies, fluxes. $\frac{1}{2}\varepsilon_0 |\vec{E}|^2$, $\vec{E}\times\vec{H}$, $T_{ij}$, ...
quadratic

Time average $\langle \cos\omega t\rangle = 0$ $\langle \cos^2\omega t\rangle = \frac{1}{2}$, ...

$$\langle AB\rangle = \left\langle \frac{1}{2}(A e^{-i\omega t} + A^* e^{+i\omega t}) \cdot \frac{1}{2}(B\, e^{-i\omega t} + B^* e^{+i\omega t})\right\rangle$$

$$= \frac{1}{4}\left\langle \cancel{\frac{1}{4} AB e^{-2i\omega t}} + A^*B + AB^* + \cancel{A^*B^* e^{+2i\omega t}}\right\rangle$$

$$= \frac{1}{4}\left(A^*B + AB^*\right) = \frac{1}{2}\mathrm{Re}(AB^*) = \frac{1}{2}\mathrm{Re}(A^*B)$$

$$\langle \vec{S}\rangle = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*) \qquad \vec{S}_{complex} = \frac{1}{2}\vec{E}\times\vec{H}^*$$

$$\langle w_e\rangle = \frac{1}{4}\mathrm{Re}(\vec{E}\cdot\vec{D}^*) \qquad w_e = \frac{1}{4}\vec{E}\cdot\vec{D}^*$$

$$\langle w_m\rangle = \frac{1}{4}\mathrm{Re}(\vec{B}\cdot\vec{H}^*) \qquad w_m = \frac{1}{2}\vec{B}\cdot\vec{H}^*$$

$$\langle \vec{J}\cdot\vec{E}\rangle = \frac{1}{2}\mathrm{Re}(\vec{J}\cdot\vec{E}^*)$$

revisit Poynting's Theorem,

$$\int_V d^3x\, \tfrac{1}{2}\vec{J}^*\cdot\vec{E} = \int_V d^3x\, \tfrac{1}{2}\Big(\vec{\nabla}\times\vec{H} - \underbrace{\frac{\partial \vec{D}}{\partial t}}_{+i\omega\vec{D}}\Big)^*\cdot\vec{E}$$

$$= \int_V d^3x\, \tfrac{1}{2}\left[(\vec{\nabla}\times\vec{H})^*\cdot\vec{E} - i\omega\vec{D}^*\cdot\vec{E}\right]$$

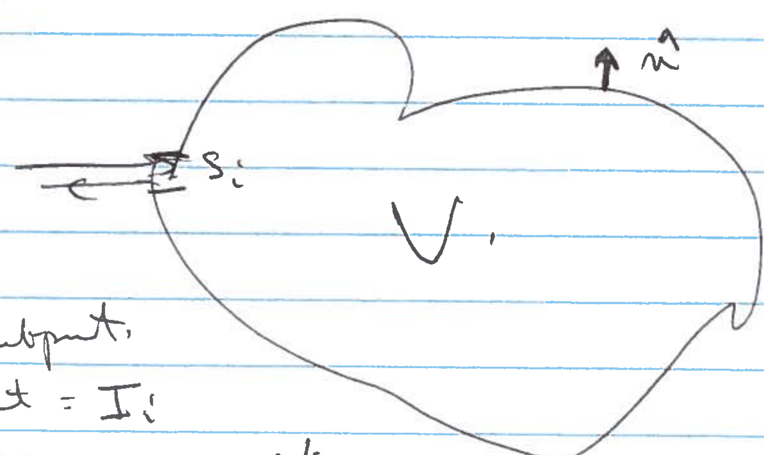
$$= \int_V d^3x\, \tfrac{1}{2}\left[-\vec{\nabla}\cdot(\vec{E}\times\vec{H}^*) + \vec{H}^*\cdot(\vec{\nabla}\times\vec{E}) - i\omega\vec{D}^*\cdot\vec{E}\right]$$

$$\uparrow\; -\frac{\partial\vec{B}}{\partial t} = +i\omega\vec{B}$$

$$= \int_V d^3x\, \tfrac{1}{2}\left[-\vec{\nabla}\cdot(\vec{E}\times\vec{H}^*) - i\omega\vec{E}\cdot\vec{D}^* + i\omega\vec{B}\cdot\vec{H}^*\right]$$

$$\int_V d^3x\left[\tfrac{1}{2}\vec{J}^*\cdot\vec{E} + 2i\omega(w_e - w_m)\right] + \oint_S d^2a\,\hat{n}\cdot\vec{S} = 0$$

(6.134).




input/output,
current = $I_i$
voltage across. = $V_i$

$\boxed{V_i = I_i Z_i}$ input impedance.

$$P_i = \int_{S_i} d^2a\,(-\hat{n})\cdot\vec{S} = \frac{1}{2}|I_i|^2 Z$$

$$= 2i\omega \int_V d^3x\,(w_e - w_m) + \int_V d^3x\,\frac{1}{2}\vec{J}^*\cdot\vec{E} + \oint_{S-S_i} d^2a\,\hat{n}\cdot\vec{S}$$

$\boxed{Z = R - iX}$ (EE: $e^{+j\omega t}$, $Z = R + jX$)

Real part
$$R = \frac{1}{\frac{1}{2}|I_i|^2}\left[\mathrm{Re}\int_V d^3x\,\frac{1}{2}\vec{J}^*\cdot\vec{E} + 2\omega\,\mathrm{Im}\int_V d^3x\,(w_e - w_m) + \mathrm{Re}\oint_{S-S_i} d^2a\,\hat{n}\cdot\vec{S}\right]$$

Imaginary
$$X = \frac{1}{\frac{1}{2}|I_i|^2}\left[2\omega\,\mathrm{Re}\int_V d^3x\,(w_m - w_e) - \mathrm{Im}\int_V d^3x\,\frac{1}{2}\vec{J}^*\cdot\vec{E}\right]$$

$\mu, \varepsilon$ real $\quad \sigma$ real $\quad \vec{J}^*\cdot\vec{E} = \sigma|\vec{E}|^2$

$$w_e = \frac{1}{4}\varepsilon|\vec{E}|^2$$

$$w_m = \frac{1}{4}\mu|\vec{H}|^2$$

$$\boxed{R = \frac{1}{|I_i|^2} \int_V d^3x \, |\vec{E}|^2}$$

$$\boxed{X = \frac{4\omega}{|I_i|^2} \int_V d^3x \, (w_e - w_m)}$$

$$W_e = \frac{1}{4} \frac{|Q|^2}{C} = \frac{1}{4} \left|\frac{I}{i\omega}\right|^2 \frac{1}{C} = \frac{1}{4} \frac{|I|^2}{\omega^2 C}$$

$$W_m = \frac{1}{4} L |I|^2$$

$$\boxed{X = \frac{\cancel{4}\omega}{\cancel{|I|^2}} \left( \frac{1}{\cancel{4}} L \cancel{|I|^2} - \frac{1}{\cancel{4}} \frac{\cancel{|I|^2}}{\omega^2 C} \right) = \omega L - \frac{1}{\omega C}}$$

Resonance. $\underline{X = 0}$. $\omega^2 = \frac{1}{LC}$

<u>Radiation flux</u>. $\langle P_{rad} \rangle = \left\langle \oint d^2a \, \hat{n} \cdot \vec{S} \right\rangle$

$$= \frac{1}{2} |I_i|^2 \, \underline{R_{rad}},$$

$\vec{E}, \vec{B}$ "vector fields"

$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$ "scalar" $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ "vector"

Q: What is a vector?

- magnitude, direction ↗ arrow.
- {set} closed under $\oplus$, $\otimes$.

Transformation: A vector is something that transforms (rotates) like a vector

Rotations preserve lengths, angles.

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$$

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|}$$

rotation preserves dot products

$\vec{x}\,' = R\vec{x}$ linear $x'_i = R_{ij} x_j$
$y'_i = R_{ik} y_k$

$\alpha\beta$

$ij$

$ab$

$\vec{x}\,' \cdot \vec{y}\,' = x'_i y'_i = \cancel{R_{ik}}\ R_{ij} x_j \cdot R_{ik} y_k$

$= (R_{ij} R_{ik}) x_j y_k = \delta_{jk} x_j y_k$.

For any $\vec{x}, \vec{y} \implies \boxed{R_{ij} R_{ik} = \delta_{jk}}$

$[R^T]_{ji} R_{ik} = \delta_{jk}$ $\boxed{R^T R = \mathbb{1}}$

"orthogonal"

$(\det R^T)(\det R) = (\det R)^2 = (\det 1) = 1$

$\boxed{\det R = \pm 1}$ $+1$: "proper"

→ $\delta_{ij}$ is a rotation $\delta_{ij}\delta_{ik} = \delta_{jk}$ ✓

<u>Near</u> identity: $R_{ij} = \delta_{ij} + \varepsilon_{ij}$

$$R_{ij} R_{ik} = (\delta_{ij} + \varepsilon_{ij})(\delta_{ik} + \varepsilon_{ik})$$

$$= \delta_{jk} + \varepsilon_{jk} + \varepsilon_{kj} + O(\varepsilon^2). = \delta_{jk}.$$

$\boxed{\varepsilon_{jk} + \varepsilon_{kj} = 0}$ $\boxed{\varepsilon_{jk} = -\varepsilon_{kj}}$ Antisymmetric

$3\times 3 \to 3$ d.f.

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

Basis

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\boxed{[J_i, J_j] = \varepsilon_{ijk} J_k}$ Lie algebra $SO(3)$.

QM. $\vec{J} \to i\hbar \vec{J}$ $[J_i, J_j] = i\hbar \, \varepsilon_{ijk} J_k$.

Infinitesimal

$$R_{ij} = \delta_{ij} + \varepsilon_1 (J_1)_{ij} + \varepsilon_2 (J_2)_{ij} + \varepsilon_3 (J_3)_{ij}$$

$$\mathbb{R} = \underline{\mathbb{1}} + \vec{\varepsilon} \cdot \vec{\underline{J}}$$

Finite $\quad \mathbb{R} = \lim_{N\to\infty} \left( \underline{\mathbb{1}} + \frac{1}{N} \vec{\theta} \cdot \vec{J} \right)^N = \exp(\vec{\theta} \cdot \vec{J})$

$$J_3 = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & \end{pmatrix} \qquad J_3^2 = \begin{pmatrix} -1 & & \\ & -1 & \\ & & \end{pmatrix} \qquad J_3^3 = -J_3, \quad J_3^4 = -J_3^2$$

$$\exp(\vec{\theta}\cdot\vec{J}) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \theta \begin{pmatrix} & -1 & \\ 1 & & \\ & & \end{pmatrix} + \frac{1}{2}\theta^2 \begin{pmatrix} -1 & & \\ & -1 & \\ & & \end{pmatrix} + \frac{1}{6}\theta^3 \begin{pmatrix} & 1 & \\ -1 & & \\ & & \end{pmatrix}$$

$$+ \frac{1}{24}\theta^4 \begin{pmatrix} 1 & & \\ & 1 & \\ & & \end{pmatrix} + \frac{1}{120}\theta^5 \begin{pmatrix} & -1 & \\ 1 & & \\ & & \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} +$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \cdots \right)$$

$$+ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \cdots \right)$$

$$= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\det = \cos^2 + \sin^2 = 1$

Rows, columns orthogonal