12/7/2015

$\vec{x}' = R\vec{x}$. $\vec{x}' \cdot \vec{y}' = \vec{x} \cdot \vec{y}$

Group . $R = R_1 R_2$.

$R^T R = \mathbb{1} \to R^{-1} = R^T$

$$R = \exp(\vec{\theta} \cdot \vec{J})$$

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\boxed{[J_i, J_j] = \varepsilon_{ijk} J_k}$ Lie algebra . SO(3)

SO(N) : pairs. $\frac{1}{2}N(N-1)$ generators

$J_3 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ $J_3^2 = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ $J_3^3 = -J_3$ $J_3^4 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

$$\exp(\theta J_3) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \theta \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} + \frac{1}{2}\theta^2 \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$$

$$+ \frac{1}{6}\theta^3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \frac{1}{24}\theta^4 \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} \left(1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \cdots\right)$$

$$+ \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \left(\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \cdots\right)$$

2.

$$\exp(\theta J_3) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$z' = z \qquad x' = x\cos\theta - y\sin\theta$

$\qquad\qquad y' = x\sin\theta + y\cos\theta$

rows columns orthogonal

$$R^{-1} = R^T \longrightarrow (\theta \to -\theta)$$

$$\log(\det M) = \log(\lambda_1\lambda_2\lambda_3) = \log\lambda_1 + \log\lambda_2 + \log\lambda_3$$

$$= \mathrm{Tr}(\ln M)$$

$$R = \exp(\vec{\theta}\cdot\vec{J})$$

$$\log\det R = \mathrm{Tr}(\vec{\theta}\cdot\vec{J}) = \theta_i\,\underbrace{(\mathrm{Tr}\,J_i)}_{0} = 0$$

$$\boxed{\det R = 1}$$ continuously connected to $\mathbb{1}$.

$$P = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \qquad P\vec{x} = -\vec{x}$$

(improper R) = P × (proper R)

~~Spezi~~

$v'_i = R_{ij} v_j$ vector.

$T'_{ijk} = R_{im} R_{jn} R_{kp} T_{mnp}$ Tensor (rank = 3)

linear sum, scalar multiplication

$\cancel{T_{ij} = A_i + B_j}$ $T_{ij} = A_i B_j$ ✓

outer product (tensor product)

Special tensors

Z_{ijk} = all zeroes $Z'_{ijk} = R_{im} R_{jn} R_{kp} Z_{ijk} = 0$

$\delta'_{ij} = R_{im} R_{jn} \delta_{mn} = R_{in} R_{jn} = (RR^T)_{ij}$

$= (RR^{-1})_{ij} = \delta_{ij}$ ← How we got here

④

$$\varepsilon'_{ijk} = R_{im} R_{jn} R_{kp} \varepsilon_{mnp}$$

(123) → det R
ijk → cyclic.

$$= (\det R) \cdot (\varepsilon_{ijk})$$

$$\varepsilon'_{ijk} = \varepsilon_{ijk}$$ for proper R.

Scalar product:

$$\vec{x}' \cdot \vec{y}' = R_{ij} x_j R_{ik} y_k = \delta_{jk} x_j y_k = \vec{x} \cdot \vec{y}$$

vector → scalar

contraction $T_{ijkl} = T_{ij}$

$$T'_{ijkl} = R_{im} R_{jn} R_{kp} R_{lq} T_{mnpq}$$

$$T'_{ij} = T'_{ijkk} = R_{im} R_{jn} \underbrace{R_{kp} R_{kq}}_{\delta_{pq}} T_{mnpq}$$

$$= R_{im} R_{jn} T_{mnpp} = R_{im} R_{jn} T_{mn}$$

Before or after contraction

$r \to r-2$

$$(\vec{x}\,' \times \vec{y}\,')_i = \varepsilon_{ijk}\, x'_j\, y'_k$$

$$= \varepsilon_{ijk}\, (R_{jm} x_m)(R_{kn} y_n)$$

$$= \delta_{ip}\, \varepsilon_{pjk}\, R_{jm}\, R_{kn}\, x_m y_n$$

$$= R_{iq} \boxed{R_{pq}\, R_{jm}\, R_{kn}\, \varepsilon_{pjk}}\, x_m y_n$$

$$= R_{iq}\, \varepsilon_{qmn}\, x_m y_n = R_{iq} (\vec{x} \times \vec{y})_q$$

For proper $\mathbb{R}$. $(\vec{x} \times \vec{y})$ transforms as a vector,

vector product is a vector

dual $A_i \rightarrow \boxed{(*A)_{jk} = \varepsilon_{ijk} A_i}$

$$(*A)_{23} = A_1$$

antisymmetric $A_{ij} = -A_{ji}$

$$\boxed{(*A)_i = \frac{1}{2} \varepsilon_{ijk} A_{jk}}$$

$$(*A)_1 = \frac{1}{2}\left(\varepsilon_{123} A_{23} + \varepsilon_{132} A_{32}\right) = A_{23}$$

$$= \frac{1}{2}\left(A_{23} - A_{32}\right)$$

$$\boxed{*(*A) = A}$$

$r \leftrightarrow d - r$

$*\phi = \phi \cdot \varepsilon_{ijk}$ $\quad r = 0$

$*A_{ijk} = \frac{1}{6} \varepsilon_{ijk} A_{ijk}$ $\quad r = 3$

3d $A_{ij} \leftrightarrow$ vector

maxwell equations

$\vec{x}$ = prototype vector.

$r = |\vec{x}|$ = scalar.

d^3x. $\quad d^3x' = \det\left|\frac{\partial x'_i}{\partial x_j}\right| d^3x = |\det \alpha|\, d^3x = d^3x$.

scalar.

Q is a scalar? tentatively. (guess).

$\rho = \frac{dQ}{dV}$ = scalar

$$\vec{E} = \int d^3x' \frac{\rho(\vec{x}')}{4\pi\epsilon_0} \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = \text{vector} \quad \vec{P}, \vec{D}.$$

$$\Phi = \int \frac{d^3x'\, \rho(\vec{x}')}{4\pi\epsilon_0 |\vec{x}-\vec{x}'|} \quad \text{scalar} \qquad \vec{E} = -\vec{\nabla}\Phi.$$

T-even $(t \to -t)$ $\qquad$ C-odd. $(q \to -q)$

$\vec{J} = \rho\vec{v}$: vector. $\quad$ P-odd

C-odd

T-odd.

$$\vec{A} = \frac{\mu_0}{4\pi}\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} \quad \text{vector}.$$

T-odd.

$\vec{B} = \vec{\nabla}\times\vec{A}$ "pseudovector" (P-even) C-odd.

$\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$ — scalar — P-even C-odd T-even

$\vec{\nabla}\cdot\vec{B} = 0$ — pseudoscalar — P-odd C-odd T-odd.

$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ — pseudo vector. — P-even C-odd T-~~odd~~ even

$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}$ — vector — P-odd C-odd T-odd

$\vec{S} = \vec{E}\times\vec{B}$ = vector.
P-odd.
T-odd
C-even

$u = \frac{1}{2}\varepsilon_0|\vec{E}|^2 + \frac{1}{2\mu_0}|\vec{B}|^2$ — scalar
P-even
T-even
C-even

$T_{ij} = \varepsilon_0(E_iE_j + \dots)$ — $r=2$ — symmetric
P-even C-even T-even.

$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ — vector
P-odd
T-even $\left(\frac{d^2}{dt^2}\right)$
C-even (q^2)