1. An end-driven full-wave thin linear antenna of length $d$ has current distribution $I = I_0 \sin kz$, with $kd = 2\pi \left(-d/2 < z < d/2\right)$.

(a) Compute the fields and power radiated per solid angle in the radiation regime but otherwise without approximation.

(b) Repeat for the leading term in the multipole expansion. Plot both angular distributions in polar coordinates.

(c) Compute (perhaps numerically) the total power radiated for the exact and multipole solutions, and find the radiation resistance.

Is there anything that bothers you about your results?

2. A system of charged particles lies within a volume $V$. The sum of the field and particle angular momentum is

$$ \mathbf{J} = \int_V d^3x \mathbf{x} \times (\epsilon_0 \mathbf{E} \times \mathbf{H}) + \sum x_i \times m \mathbf{v}_i $$

(a) Show that the rate of change of $\mathbf{J}$ is

$$ \frac{d\mathbf{J}}{dt} = \int_S d^2a \left[ \epsilon_0 \hat{n} \cdot \mathbf{E} \times \mathbf{x} + \mu_0 \hat{n} \cdot \mathbf{H} \times \mathbf{x} \right] + \int_S d^2a \hat{n} \times \mathbf{x} \left( \frac{1}{2} \epsilon_0 |\mathbf{E}|^2 + \frac{1}{2} \mu_0 |\mathbf{H}|^2 \right) $$

where $S$ is any surface that encloses $V$.

(b) Use the fields of an electric dipole to compute $\langle d\mathbf{J}/dt \rangle$ through a distant sphere when the particle motion generates an oscillating dipole moment $Re[\mathbf{p} e^{-i\omega t}]$. Show that

$$ \langle d\mathbf{J}/dt \rangle = \frac{k^3 \text{Im}[\mathbf{p}^* \times \mathbf{p}]}{12\pi\epsilon_0}.$$

(c) What is the ratio of angular momentum flux to energy flux in electric dipole radiation? Evaluate for linearly polarized radiation, with $\mathbf{p} = p \hat{z}$, and for circularly polarized radiation, with $\mathbf{p} = p (\hat{x} \pm i\hat{y})$.

(d) What is the angular momentum lost for magnetic dipole radiation? (Hint: Don’t do a calculation.)
3. An antisymmetric “quadrupole” $\tilde{Q}_{ij}$ has the form

$$
\tilde{Q} = \begin{pmatrix}
0 & \tilde{q}_3 & -\tilde{q}_2 \\
-\tilde{q}_3 & 0 & \tilde{q}_1 \\
\tilde{q}_2 & -\tilde{q}_1 & 0
\end{pmatrix}
$$

(a) Let $\tilde{q}_1 = \tilde{q}_2 = 0$ and $\tilde{q}_3 = q_0$. Find the angular distribution and total power radiated following Jackson’s quadrupole procedure.

(b) Let $\tilde{q}_1 = q_0/\sqrt{2}$, $\tilde{q}_2 = iq_0/\sqrt{2}$ and $\tilde{q}_3 = 0$. Find the angular distribution and total power radiated following Jackson’s quadrupole procedure.

(c) Recalling the definition of the dual of a vector (see class notes from 7 December in the Fall; Mathworld gives you one half of the relation, but other online sources are not much help), use the relation between $\tilde{Q}_{ij}$ and $\tilde{q}$ to write the angular distribution and radiated power in terms of an arbitrary $\tilde{q}$. What does it remind you of?

4. Three charges $+q$ rotate $120^\circ$ apart about the $z$-axis along a circle of radius $a$.

(a) Compute the instantaneous electric dipole, magnetic dipole, and electric quadrupole moments. Write as the real part of a complex amplitude times $e^{-i\omega t}$.

(b) What is the leading contribution to the radiated power? What is the frequency of the emitted radiation? Compute the angular distribution and total power.

5. Gravitational wave event GW 150914 induced a strain (relative distortion) of the earth of order $10^{-21}$ at frequencies of order $10^2$ Hz. Meanwhile, the earth’s surface has permanent distortions (“mountains”) of order $\Delta R \sim$ (few to several) km which rotate at a rate of once a day. Estimate (within a factor of a million or so is OK) how long it takes the everyday rotation of the earth to emit as much gravitational wave energy as that induced by GW 150914.