# 't Hooft-Polyakov Magnetic Monopoles

## I.   GENERALITIES

Magnetic monopoles arise naturally when considering non-singular, finite energy solutions to non-Abelian gauge theories in three-dimensional space [1, 2]. Let

$$\mathcal{L} = \mathcal{D}_\mu \boldsymbol{\phi} \cdot \mathcal{D}^\mu \boldsymbol{\phi} - \frac{1}{4}\, \boldsymbol{G}_{\mu\nu} \cdot \boldsymbol{G}^{\mu\nu} - V(\boldsymbol{\phi}), \tag{1}$$

where $\mathcal{D}^\mu$ is a combination of $\nabla^\mu$ and the gauge potentials $\boldsymbol{W}^\mu$; assume $V(\boldsymbol{\phi})$ has minimum value zero. It is always possible to choose a gauge in which the time components $W_0^a$ vanish (for any given solution, we can find a gauge transformation which takes it to a solution with $W_a^0 = 0$ [2]). In such a gauge, we have $\mathcal{D}_0\boldsymbol{\phi} = \dot{\boldsymbol{\phi}}$, $\boldsymbol{G}_{0i} = \dot{\boldsymbol{W}}_i$, and the energy takes the form

$$E = T + V = \int d^3x \left[ |\dot{\boldsymbol{\phi}}|^2 + \frac{1}{2}|\dot{\boldsymbol{W}}^j|^2 + |\mathcal{D}^i\boldsymbol{\phi}|^2 + \frac{1}{4}|\boldsymbol{G}^{ij}|^2 + V(\boldsymbol{\phi}) \right]. \tag{2}$$

For finite energy as $r = |\boldsymbol{x}| \to \infty$ we need

$$V(\boldsymbol{\phi}) \to 0, \qquad |\mathcal{D}^i\boldsymbol{\phi}| = o(r^{-3/2}). \tag{3}$$

We would like to then conclude that $\boldsymbol{\phi}$ must approach a limit which is a zero of the potential, but such a statement is not gauge invariant: given a gauge where it is true, we may always perform a gauge transformation which is an oscillatory function of $r$ and thus will make $\boldsymbol{\phi}$ oscillate at infinity. However, at any given instant it is possible to find a gauge in which the radial components $W_a^r = \hat{r}^i W_i^a$ vanish for all $r$ greater than some nonzero radius. In this gauge, equation (3) implies $|\partial\boldsymbol{\phi}/\partial r| = o(r^{-3/2})$, and so indeed

$$\boldsymbol{\phi}_\infty(\hat{\boldsymbol{r}}) = \lim_{r\to\infty} \boldsymbol{\phi}(r\hat{\boldsymbol{r}}) \tag{4}$$

exists for each direction $\hat{\boldsymbol{r}}$. The map $\boldsymbol{\phi}_\infty(\hat{\boldsymbol{r}})$ takes the "two-sphere" $S^2$ (directions in physical space) to the set $\mathcal{M}_0$ of zeroes of $V$ (directions in the internal space of $\boldsymbol{\phi}$, or the coset space $G/H$). As a result of the equations of motion, we expect that $\boldsymbol{\phi}_\infty$ varies continuously with time, and so its topological characteristics will be preserved.

Can $\boldsymbol{\phi}_\infty(\hat{\boldsymbol{r}})$ depend nontrivially on $\hat{\boldsymbol{r}}$? It would seem that if so, then as $r \to \infty$ the transverse components of $\nabla_i\boldsymbol{\phi}$ will behave as $1/r$, a behavior which leads to a divergent energy in equation (2). However, the quantity that appears in the energy is not the gradient but the covariant gradient. Thus, in order to have finite energy we must have

$$|\hat{\boldsymbol{r}} \wedge \nabla_i\boldsymbol{\phi}| = O(r^{-1}), \qquad |\hat{\boldsymbol{r}} \wedge \boldsymbol{W}_i| = O(r^{-1}), \qquad |\hat{\boldsymbol{r}} \wedge \mathcal{D}_i\boldsymbol{\phi}| = o(r^{-3/2}); \tag{5}$$

that is, a seemingly "delicate" cancellation has to occur. This, it turns out, is exactly what happens: the variation of $\boldsymbol{\phi}$ in space are cancelled by a group transformation that leaves vanishing $\mathcal{D}_i\boldsymbol{\phi}$. The finiteness of the energy connects the large-$r$ behavior of the gauge fields with that of the scalar fields,

$$\lim_{r\to\infty} rW_i^a T_a\boldsymbol{\phi}_\infty(\hat{\boldsymbol{r}}) = -\frac{i}{e}\nabla_i\boldsymbol{\phi}. \tag{6}$$

In particular, this means that the gauge potentials must go like $1/r$ at large $r$, which we recognize as characteristic of the fields of point charges. This means that the field strengths go as $1/r^2$, their squares as $1/r^4$, and the energy integral is convergent at large $r$ for up to three space dimensions.

## II.   THE 'T HOOFT-POLYAKOV MONOPOLE

Magnetic monopoles in non-Abelian gauge theories were discovered by 't Hooft[3] and Polyakov [4]. Consider an SU(2) model in which the scalar field $\phi$ transforms according to the adjoint representation (a vector in the internal space), with potential

$$V(\phi) = \frac{\lambda}{4}(|\phi|^2 - a^2)^2. \tag{7}$$

The adjoint fields $\boldsymbol{W}^\mu$ and $\boldsymbol{G}^{\mu\nu} = \nabla^\mu \boldsymbol{W}^\nu - \nabla^\nu \boldsymbol{W}^\mu - e\boldsymbol{W}^\mu \wedge \boldsymbol{W}^\nu$ are also vectors, and the covariant derivative is $\mathcal{D}^\mu \phi = \nabla^\mu \phi - e\boldsymbol{W}^\mu \wedge \phi$. The Lagrangian density is

$$\mathcal{L} = \mathcal{D}_\mu \phi \cdot \mathcal{D}^\mu \phi - \frac{1}{4} \boldsymbol{G}_{\mu\nu} \cdot \boldsymbol{G}^{\mu\nu} - \frac{\lambda}{4}(|\phi|^2 - a^2)^2, \tag{8}$$

with equations of motion

$$\mathcal{D}_\mu \mathcal{D}^\mu \phi = -\lambda \phi \left( |\phi|^2 - a^2 \right), \qquad \mathcal{D}_\nu \boldsymbol{G}^{\mu\nu} = e\phi \wedge \mathcal{D}^\mu \phi = -\boldsymbol{j}^\mu. \tag{9}$$

We are interested in cases where $\phi$ has a nontrivial angular behavior at infinity. In this model, the set of zeroes (minima) of the potential,

$$\mathcal{M}_0 = \{\phi : |\phi|^2 = a^2\}, \tag{10}$$

is a two-dimensional sphere. A candidate possible nontrivial angular behavior is

$$\phi_\infty(\hat{\boldsymbol{r}}) = \lim_{r \to \infty} \phi(r\hat{\boldsymbol{r}}) = a\hat{\boldsymbol{r}}. \tag{11}$$

An *Ansatz* which has this behavior is

$$\phi = \hat{\boldsymbol{r}} \frac{H(aer)}{er}, \qquad W_a^i = \epsilon_{aij} \hat{\boldsymbol{x}}^j \frac{1 - K(aer)}{er}, \qquad W_a^0 = 0. \tag{12}$$

The dimensionless radius is $\xi = aer$. The energy in the fields is

$$E = \frac{4\pi a}{e} \int_0^\infty \frac{d\xi}{\xi^2} \left[ \xi^2 \left( \frac{dK}{d\xi} \right)^2 + \frac{1}{2} \left( \xi \frac{dH}{d\xi} - H \right)^2 + \frac{1}{2}(K^2 - 1)^2 + K^2 H^2 + \frac{\lambda}{4e^2}(H^2 - \xi^2)^2 \right] \tag{13}$$

with equations of motion

$$\xi^2 \frac{d^2 K}{d\xi^2} = KH^2 + K(K^2 - 1), \tag{14}$$

$$\xi^2 \frac{d^2 H}{d\xi^2} = 2K^2 H + \frac{\lambda}{e^2} H(H^2 - \xi^2) \tag{15}$$

(these equations in fact follow from variation of the energy in [13]). The boundary conditions are $K - 1 < O(\xi)$ and $H < O(\xi)$ as $\xi \to 0$ and $K \to 0$, $H \sim \xi$ sufficiently fast as $\xi \to \infty$. "Sufficiently fast" is not a problem; with $H \sim \xi$, the first equation becomes $d^2K/d\xi^2 = K$ as $\xi \to \infty$, and the desired solution will approach zero as $e^{-\xi}$.

The full solution to these equations is difficult; for the general case there is not even a numerical result published. For $\lambda \to 0$ (retaining the condition $|\phi|^2 = a^2$ at infinity), Prasad and Sommerfield [5] find an analytic solution,

$$K = \xi \operatorname{csch} \xi, \qquad H = \xi \coth \xi - 1, \tag{16}$$

for which the integral in (13) is exactly 1. In general, the energy (or rest mass) is

$$M = E = \frac{4\pi a}{e} f\left(\frac{\lambda}{e^2}\right), \tag{17}$$

where the function $f$ is of order 1. Some other published values are

$$f(0.1) = 1.1 \ ^{[3]}, \qquad f(0.5) = 1.42 \ ^{[9]}, \qquad f(10) = 1.44 \ ^{[3]}, \tag{18}$$

but my guess is that not all of these are accurate.

For $\lambda/e^2 \gg 1$, we expect $H \to \xi$, except for a small region near $\xi = 0$. This can be made quantitative: if we write $H = \xi(1 - h)$, then

$$\frac{d^2h}{d\xi^2} + \frac{2}{\xi}\frac{dh}{d\xi} = \frac{2\lambda}{e^2}h(1 - h)(1 - \tfrac{1}{2}h) \to \frac{2\lambda}{e^2}h; \tag{19}$$

and thus $h$ falls exponentially $h \sim e^{-\xi/\epsilon}$, with scale $\epsilon = \sqrt{2\lambda/e^2} = M_H/M_W$ [the solution for small $h$ is in fact a modified Bessel function, $h = k_0(\xi/\epsilon)$].

Using the variational principle, any pair of functions $H$, $K$ we make up places an upper bound on the energy. For large $\lambda$ we expect to drive $H \to \xi$. With $H = \xi$, the PS function $K = \xi \operatorname{csch} \xi$ gives $f = \frac{1}{2} + \zeta(2) = (3 + \pi^2)/6 = 2.14493$. The function $K = \operatorname{sech} \xi$, another pure guess with the right boundary behaviors, gives $f = 1.81008$, a quite reasonable upper bound, since fixing $H = \xi$ and solving for the best $K$ gives $f = 1.786609$ [6–8]. The values 1 and 1.786609 bracket the possible values, since the function $f(\lambda)$ is monotonic,

$$\frac{df}{d\lambda} = \frac{\partial f}{\partial \lambda} = \int \frac{d\xi}{\xi^2} \frac{1}{4e^2} \left(H^2 - \xi^2\right)^2 \geq 0, \tag{20}$$

(the first equality since the functions $H$ and $K$ are solutions to $\delta f/\delta H = \delta f/\delta K = 0$). For small and large values of $\epsilon$, Kirkman and Zachos [8] find

$$f \to \begin{cases} 1 + \epsilon/2 + \cdots & (\epsilon \to 0) \\ 1.787 - 2.228/\epsilon + \cdots & (\epsilon \to \infty) \end{cases} \tag{21}$$

where again $\epsilon = \sqrt{2\lambda/e^2}$.

The plot (next page) shows the dimensionless integral $f$ as a function of $\lambda/e^2$. Horizontal long-dash lines the values $f = 1$ ($\lambda \to 0$) and $f = 1.786609$ ($\lambda \to \infty$); short dash lines show the Kirkman and Zachos first corrections for small and large $\epsilon$. Open symbols connected by dotted lines show variational upper limits found from $K = \xi \operatorname{csch} \xi$ (circles) and $K = \operatorname{sech} \xi$ (squares). Filled symbols show the values in (18). A combination of the dashed lines and the lower envelope of the open points represents an approximation to $f(\lambda)$. If anyone obtains a better numerical result, please let me know.
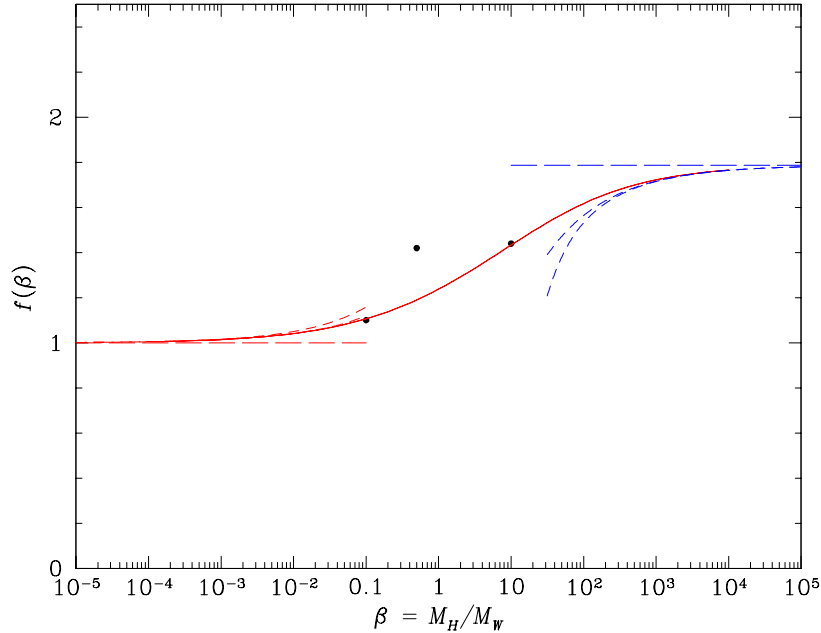
## III.  ELECTROMAGNETISM AND MAGNETIC CHARGE

If the scalar field tends to $|\boldsymbol{\phi}| = a$ at infinity, the $\boldsymbol{W}^\mu \wedge \boldsymbol{\phi}$ terms in the Lagrangian become

$$\mathcal{L} = \cdots + e^2 a^2 \left[ \boldsymbol{W}_\mu \cdot \boldsymbol{W}^\mu - \frac{1}{a^2} (\boldsymbol{W}_\mu \cdot \boldsymbol{\phi})(\boldsymbol{W}^\mu \cdot \boldsymbol{\phi}) \right].$$

This gives the gauge bosons transverse to $\boldsymbol{\phi}$ masses $M_t^2 = M_W^2 = e^2 a^2$, but leaves massless the mode along the direction of $\boldsymbol{\phi}$ (the vector fields associated with elements of the Lie algebra which generate transformations that do not preserve $\boldsymbol{\phi}_0$ acquire a mass, while the vector fields associated with the generators which leave $\boldsymbol{\phi}_0$ fixed [the "little group" of $\boldsymbol{\phi}_0$] remain massless). It is tempting to interpret this massless mode as electromagnetism. Consider the "electromagnetic" field tensor,

$$F^{\mu\nu} = \frac{\boldsymbol{\phi}}{a} \cdot \boldsymbol{G}^{\mu\nu}. \tag{22}$$

Using the symmetry of the triple scalar product, $(\boldsymbol{W}_\lambda \wedge \boldsymbol{\phi}) \cdot \boldsymbol{G}^{\mu\nu} = -\boldsymbol{\phi} \cdot (\boldsymbol{W}_\lambda \wedge \boldsymbol{G}^{\mu\nu})$, derivatives of this tensor are gauge invariant,

$$\nabla_\lambda F^{\mu\nu} = (\frac{1}{a} \nabla_\lambda \boldsymbol{\phi}) \cdot \boldsymbol{G}^{\mu\nu} + \frac{\boldsymbol{\phi}}{a} \cdot (\nabla_\lambda \boldsymbol{G}^{\mu\nu}) = (\frac{1}{a} \mathcal{D}_\lambda \boldsymbol{\phi}) \cdot \boldsymbol{G}^{\mu\nu} + \frac{\boldsymbol{\phi}}{a} \cdot (\mathcal{D}_\lambda \boldsymbol{G}^{\mu\nu}). \tag{23}$$

We can write

$$\nabla_\nu F^{\mu\nu} = -j^\mu, \qquad \nabla_\nu F^{*\mu\nu} = -k^\mu, \tag{24}$$

where, using the equations of motion and the Bianchi identities,

$$j^\mu = -(\frac{1}{a} \mathcal{D}_\nu \boldsymbol{\phi}) \cdot \boldsymbol{G}^{\mu\nu} + \frac{\boldsymbol{\phi}}{a} \cdot \boldsymbol{j}^\mu, \qquad k^\mu = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\frac{1}{a} \mathcal{D}_\nu \boldsymbol{\phi}) \cdot \boldsymbol{G}_{\rho\sigma} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \nabla_\nu F_{\rho\sigma}. \tag{25}$$

At infinity, where $\mathcal{D}_\lambda \phi \to 0$ exponentially fast, Maxwell's equations are satisfied with $k^\mu = 0$ and $j^\mu$ the appropriate component of the SU(2) current $\boldsymbol{j}^\mu$. However, flux integrals at infinity show a nonzero magnetic charge inside.

Having identified the magnetic field, we can calculate the magnetic flux and thus the magnetic charge of the 't Hooft-Polyakov solution. In general, in a region where $|\phi| = a$, the condition

$$\mathcal{D}_\mu \phi = \nabla_\mu \phi - e\boldsymbol{W}_\mu \wedge \phi \tag{26}$$

is satisfied by

$$\boldsymbol{W}_\mu = \frac{1}{a^2 e}\phi \wedge \nabla_\mu \phi + \Lambda_\mu \phi, \tag{27}$$

where $\Lambda_\mu$ is an arbitrary function. This gives

$$\boldsymbol{G}_{\mu\nu} = \frac{1}{a^2 e}\nabla_\mu \phi \wedge \nabla_\nu \phi + (\nabla_\mu \Lambda_\nu - \nabla_\nu \Lambda_\mu)\phi, \tag{28}$$

or

$$F_{\mu\nu} = \frac{1}{a}\phi \cdot \boldsymbol{G}_{\mu\nu} = \frac{1}{a^2 e}\phi \cdot (\nabla_\mu \phi \wedge \nabla_\nu \phi) + a(\nabla_\mu \Lambda_\nu - \nabla_\nu \Lambda_\mu). \tag{29}$$

For the 't Hooft-Polyakov solution, $\Lambda_\mu = 0$, $G_{0i} = 0$, and

$$(G_a)_{ij} = \frac{\epsilon_{ijk}\hat{\boldsymbol{x}}^k \hat{\boldsymbol{x}}^a}{er^2}, \qquad F_{ij} = \frac{\epsilon_{ijk}\hat{\boldsymbol{x}}^k}{er^2} \tag{30}$$

(this also follows from eq. [12] with $K \to 0$). This is a radial magnetic field,

$$B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk} = \frac{1}{er^2}\hat{\boldsymbol{r}}_i \tag{31}$$

with total magnetic charge

$$g = \lim_{r\to\infty} \oint \boldsymbol{B} \cdot d\boldsymbol{S} = -\frac{4\pi}{e}. \tag{32}$$

In general, if $\boldsymbol{n}(\hat{\boldsymbol{r}})$ is a nonsingular unit vector function of $\theta$, $\phi$, the asymptotic forms

$$\phi = a\boldsymbol{n}, \qquad \boldsymbol{W}^\mu = \frac{1}{e}\boldsymbol{n} \wedge \nabla^\mu \boldsymbol{n}, \qquad \boldsymbol{G}^{\mu\nu} = \frac{1}{e}\nabla^\mu \boldsymbol{n} \wedge \nabla^\nu \boldsymbol{n} \tag{33}$$

are exact solutions to the equations of motion. The magnetic charge for such a configuration is

$$g = \lim_{r\to\infty} \oint \boldsymbol{B} \cdot d\boldsymbol{S} = -\frac{1}{2}\oint dS^i \epsilon_{ijk}F^{jk} = -\frac{4\pi}{e}N, \tag{34}$$

where

$$N = \frac{1}{4\pi}\oint dS^i \frac{1}{2}\epsilon_{ijk}\,\boldsymbol{n} \cdot (\nabla^j \boldsymbol{n} \wedge \nabla^k \boldsymbol{n}). \tag{35}$$

This $N$ is an integer, the number of times the unit sphere in real space is mapped onto the sphere $|\phi| = a$ by $\phi_\infty(\hat{\boldsymbol{r}}) = \boldsymbol{n}a$. The unit of electric charge is $q = \frac{1}{2}e$, thus satisfying the Dirac condition

$$\frac{qg}{4\pi} = \frac{1}{2}N \tag{36}$$

## IV. TOPOLOGICAL CURRENTS AND THE BOGOMOLNY BOUND

The 't Hooft-Polyakov solutions possess a quantized magnetic charge, related to a topological property (winding number) of $\boldsymbol{\phi}_\infty(\hat{\boldsymbol{r}})$. Since $\boldsymbol{\phi}_\infty(\hat{\boldsymbol{r}})$ evolves continuously with time, this charge is conserved. Much as soliton number in the sine-Gordon model, this quantity, conserved for topological reasons, can be obtained as a charge from a current whose divergence vanishes independently of the dynamics; the current $k^\mu$ in (25) obeys $\nabla_\mu k^\mu = 0$ from the antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$.

This provides an alternative way to determine the minimum mass of a monopole: Start with the inequality

$$\left| \mathcal{D}_i\boldsymbol{\phi} \pm \frac{1}{2}\epsilon_{ijk}\boldsymbol{G}^{jk} \right|^2 \geq 0, \tag{37}$$

so

$$\frac{1}{2}|\mathcal{D}_i\boldsymbol{\phi}|^2 + \frac{1}{4}|\boldsymbol{G}^{jk}|^2 \geq \pm\frac{1}{2}\epsilon_{ijk}\mathcal{D}^i\boldsymbol{\phi}\cdot\boldsymbol{G}^{jk}. \tag{38}$$

Thus, for a time-independent configuration,

$$\begin{aligned} E &= \int d^3x \left[ |\mathcal{D}^i\boldsymbol{\phi}|^2 + \frac{1}{4}|\boldsymbol{G}^{ij}|^2 + V(\boldsymbol{\phi}) \right] \\ &\geq \int d^3x \left[ \frac{1}{2}|\mathcal{D}^i\boldsymbol{\phi}|^2 + \frac{1}{4}|\boldsymbol{G}^{ij}|^2 \right] \geq \pm\int d^3x \frac{1}{2}\epsilon_{ijk}\mathcal{D}_i\boldsymbol{\phi}\cdot\boldsymbol{G}^{jk} = \pm ga, \end{aligned} \tag{39}$$

so $E \geq |g|a$. For a solution with magnetic charge $g = 4\pi N/e$, this says

$$M_N \geq \frac{4\pi|N|}{e} = |N|\frac{M_W}{4\alpha}.$$

The Prasad-Sommerfield solution saturates this bound for $N = 1$. The fact that the Bogomolny bound is satisfied for the Prasad-Sommerfield solution means that it must satisfy the equation

$$\mathcal{D}_i\boldsymbol{\phi} = \frac{1}{2}\epsilon_{ijk}\boldsymbol{G}^{jk}. \tag{40}$$

This relation plus the Bianchi identities imply the equations of motion for a static solution, thus reducing the equations of motion to first order equations. This provides a systematic way of arriving at the Prasad-Sommerfield solution.

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