2/9/2017 Hydrodynamics.

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0.$$

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right) = -\vec{\nabla}p + \eta\frac{\partial}{\partial x_j}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{v}\right]$$

w/o viscosity. linearize about $(\rho=\rho_0)$ $(\vec{v}=\vec{0})$

$\rho = \rho_0 + \delta\rho \qquad \vec{v} = \vec{0} + \delta\vec{v}$

$$\frac{\partial}{\partial t}(\cancel{\rho_0} + \delta\rho) + \vec{\nabla}\cdot\left[(\rho_0 + \cancel{\delta\rho})\vec{\nabla}\cdot\delta\vec{v}\right] = 0$$

$$\boxed{\frac{\partial}{\partial t}(\delta\rho) + \rho_0\vec{\nabla}\cdot(\delta\vec{v}) = 0}$$

$p(\rho_0) + \left(\frac{dp}{d\rho}\right)_0\cdot\delta\rho$

$$(\rho_0 + \cancel{\delta\rho})\left[\frac{\partial}{\partial t}(\delta\vec{v}) + (\cancel{(\delta\vec{v})\cdot\vec{\nabla}})\,\delta\vec{v}\right] = -\vec{\nabla}p$$

$$\boxed{\rho_0\frac{\partial}{\partial t}(\delta\vec{v}) = -\left(\frac{dp}{d\rho}\right)_0\vec{\nabla}(\delta\rho).}$$

$\frac{\partial}{\partial t}$(1) $\qquad \frac{\partial^2}{\partial t^2}(\delta\rho) = -\rho_0\frac{\partial}{\partial t}(\vec{\nabla}\cdot\delta\vec{v})$

$\vec{\nabla}\cdot$(2) $\qquad \rho_0\vec{\nabla}\cdot\left(\frac{\partial}{\partial t}(\delta\vec{v})\right) = -\left(\frac{dp}{d\rho}\right)_0\cdot\nabla^2(\delta\rho)$

$$\boxed{\nabla^2(\delta\rho) - \frac{1}{(dp/d\rho)_0}\frac{\partial^2}{\partial t^2}(\delta\rho) = 0}$$

$c_s^2 = \frac{dp}{d\rho}.$

$c_s^2 = \left(\frac{dp}{d\rho}\right)_0 . \quad \left(\nabla^2 - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right)\delta\rho = 0.$

Wave equation.

Ideal gas $\quad p = \frac{\rho kT}{m}.$

Isothermal $\quad \frac{dp}{d\rho} = \frac{kT}{m}.$

Adiabatic $\quad pV^\gamma = \text{constant} \quad \left| \frac{dp}{d\rho} = \frac{\gamma p}{\rho} = \frac{\gamma kT}{m} \right.$

$p = \text{constant}\cdot\rho^\gamma$

N_2, O_2. diatomic $\quad \gamma = \frac{7}{5}$

$m = (0.8)(28) + (0.2)(32) = 28.8\, m_p.$

$kT = (1\,eV)\left(\frac{T}{11{,}600\,K}\right) \simeq \frac{1}{40}\,eV \quad \left(\begin{array}{c}17C\\ = 62F\end{array}\right)$

$c_s^2 = \frac{(1.4)\left(\frac{1}{40}eV\right)}{(28.8)(0.938\cdot 10^9\,GeV/c^2)} = 1.295\times10^{-12}\,c^2$

$c_s = 1.138\times10^{-6}\cdot c = 340\, m/s$

$= 1100\, ft/s$

3

<u>Wave</u>: $\delta\rho = \rho_1 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$$\delta\vec{v} = \vec{v}_1 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\boxed{\begin{aligned} -i\omega\rho_1 + \rho_0(i\vec{k}\cdot\vec{v}_1) &= 0 \\ \rho_0(-i\omega\vec{v}_1) &= -c_s^2\, i\vec{k}\rho_1 \end{aligned}}$$

$$-i\omega^2\rho_1 = -i\omega\rho_0(\vec{k}\cdot\vec{v}_1)$$

$$= \vec{k}\cdot(-i\omega\rho_0\vec{v}_1) = -c_s^2\, ik^2\rho_1$$

$$\omega^2 = c_s^2 k^2 \qquad \frac{\omega}{k} = \frac{d\omega}{dk} = c_s$$

$$-i\omega^2\cancel{\rho_0}\vec{v}_1 = -c_s^2\, i\omega\vec{k}\rho_1$$

$$= c_s^2\vec{k}(-i\omega\rho_1) \cancel{= -\rho_0 i(\vec{k}\cdot\vec{v}_1)}$$

$$\cancel{= c_s^2\vec{k}(-i\omega\rho_0\vec{k}\cdot\vec{v}_1)}$$

$$= c_s^2\vec{k}(-\cancel{\rho_0}\, i\vec{k}\cdot\vec{v}_1)$$

$$\boxed{\omega^2\vec{v}_1 = c_s^2\vec{k}(\vec{k}\cdot\vec{v}_1)} \qquad \omega^2(\vec{k}\cdot\vec{v}_1) = c_s^2k^2(\vec{k}\cdot\vec{v}_1)$$

$$\vec{k}\times\vec{v}_1 = 0$$

$\vec{v}_1 \parallel \hat{k}$ longitudinal $\qquad -i\omega\rho_1 = \rho_0 v_1 \hat{k}\cdot\vec{k} \qquad v_1 = \frac{\rho_1 c_s}{\rho_0}\hat{k}$

4

Add. $\vec{E}, \vec{B}$

$\vec{\nabla}\cdot\vec{E} = \rho_e/\varepsilon_0 \simeq 0$ (MHD regime).

$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ ✓

$\vec{E}$ not "dynamic"

$\vec{\nabla}\times\vec{B} = \mu\vec{J} + \cancel{\mu\varepsilon\frac{\partial \vec{E}}{\partial t}}^{\,0\ (\text{MHD})} = \mu\vec{J} = \mu\sigma\left(\vec{E} + \vec{v}\times\vec{B}\right)$

$$\hookrightarrow \boxed{\vec{E} = \frac{1}{\mu\sigma}\left(\vec{\nabla}\times\vec{B}\right) - \vec{v}\times\vec{B}}$$

"constrained"

Faraday $\vec{\nabla}\times\vec{E} = \frac{1}{\mu\sigma}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) - \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right) = -\frac{\partial\vec{B}}{\partial t}$

$\hookrightarrow \nabla\cancel{(\nabla\cdot B)} - \nabla^2\vec{B}$

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{v}\times\vec{B}\right) + \frac{1}{\mu\sigma}\nabla^2\vec{B}} \qquad (7.68)$$

$\hookrightarrow$ dissipation

ignore to start.

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{v}\right) = 0$$

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right) = -\vec{\nabla}p + \vec{J}\times\vec{B}$$

⑤

Conservative. $\vec{J} = \frac{1}{\mu}(\vec{\nabla}\times\vec{B})$

$$\vec{J}\times\vec{B} = -\frac{1}{\mu}\vec{B}\times(\vec{\nabla}\times\vec{B})$$

$$\vec{B}\times(\vec{\nabla}\times\vec{B}) = B_j \nabla_i B_j - B_j \nabla_j B_i$$

$$= \vec{\nabla}\left(\tfrac{1}{2}B^2\right) - (\vec{B}\cdot\vec{\nabla})\vec{B}$$

magnetic pressure $= \frac{B^2}{2\mu}$

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right) = -\vec{\nabla}p - \vec{\nabla}\left(\frac{B^2}{2\mu}\right) + \frac{1}{\mu}(\vec{B}\cdot\vec{\nabla})\vec{B}$$

(JDJ: equation following (7.69) $\cancel{\tfrac{1}{2}} \to \left(\frac{1}{\mu}\right)$)

Lots of solutions..

scenario: $\rho = \rho_0 + \delta\rho$, $\vec{v} = 0 + \delta\vec{v}$, $\vec{B} = \vec{B}_0 + \delta\vec{B}$

$$\frac{\partial}{\partial t}(\delta\rho) + \rho_0(\vec{\nabla}\cdot\delta\vec{v}) = 0$$

$$\rho_0\frac{\partial}{\partial t}(\delta\vec{v}) = -c_s^2\,\vec{\nabla}(\delta\rho) - \frac{1}{\mu}\vec{B}_0\times(\vec{\nabla}\times\delta\vec{B}).$$

$$\frac{\partial}{\partial t}(\delta\vec{B}) = \vec{\nabla}\times(\delta\vec{v}\times\vec{B}_0)$$

scalar + 2 vector

$\vec{\nabla}$ ①

$$\vec{\nabla}\left(\frac{\partial}{\partial t}\delta\rho\right) = -\rho_0\,\vec{\nabla}(\vec{\nabla}\cdot\delta\vec{v})$$

$\frac{\partial}{\partial t}$ ②

$$\rho_0\frac{\partial^2}{\partial t^2}(\delta\vec{v}) = -c_s^2\frac{\partial}{\partial t}(\vec{\nabla}\delta\rho) - \frac{1}{\mu}\vec{B}_0\times\left(\vec{\nabla}\times\frac{\partial}{\partial t}\delta\vec{B}\right)$$

③

$$\frac{\partial}{\partial t}(\delta\vec{B}) = \vec{\nabla}\times(\delta\vec{v}\times\vec{B}_0)$$

$$\rho_0\frac{\partial^2}{\partial t^2}(\delta\vec{v}) = c_s^2\cdot\rho_0\,\vec{\nabla}(\vec{\nabla}\cdot\delta\vec{v}) - \frac{1}{\mu}\vec{B}_0\times\left(\vec{\nabla}\times\frac{\partial}{\partial t}(\delta\vec{B})\right)$$

$$-\frac{1}{\mu}\vec{B}_0\times\left[\vec{\nabla}\times\left(\vec{\nabla}\times(\delta\vec{v}\times B_0)\right)\right]$$

let $\vec{V}_A = \frac{\vec{B}_0}{\sqrt{\mu\rho_0}}$ "Alfvén velocity"

Hannes Alfvén

$$\boxed{\begin{aligned}\frac{\partial^2}{\partial t^2}(\delta\vec{v}) &= c_s^2\,\vec{\nabla}(\vec{\nabla}\cdot\delta\vec{v})\\ &\quad -\vec{V}_A\times\left[\vec{\nabla}\times\left(\vec{\nabla}\times(\delta\vec{v}\times\vec{V}_A)\right)\right]\end{aligned}}$$

7.72 ※

$\delta\vec{v} = \vec{v}_1\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

~~$\rho = \rho_0 + \delta\rho$~~ $\delta\rho = \rho_1\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$\delta\vec{B} = \vec{B}_1\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$

$-\omega^2\vec{v}_1 = M\vec{v}$

HW.