2/16/2018 $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} \longrightarrow (\nabla^2 + k^2)\vec{A} = -\mu_0 \vec{J}$

$e^{-i\omega t}$

Green's function

$$\vec{A} = \frac{\mu_0}{4\pi}\int d^3x' \, \vec{J}(\vec{x}') \, \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \quad e^{-i\omega t}$$

outgoing waves

(retarded time)

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x' \, \vec{J}(\vec{x}') \, e^{-ik\hat{r}\cdot\vec{x}'} \qquad r \gg d$$

$kd \ll 1$ long wavelength. $e^{-ik\hat{r}\cdot\vec{x}'} \simeq 1$.

integral $\to$ constant, $\int d^3x \, \vec{J} = -i\omega\vec{p}$

2

$$\boxed{\vec{A}_{E.D.} = -i\omega \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \vec{p}} \quad (9.16).$$

charge conserved → can't have oscillating Q

[GR : mass, momentum conserved.
center of mass fixed → leading Quadrupole]

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla}\times\vec{A} = \frac{1}{\cancel{\mu_0}}\left(\hat{r}\frac{d}{dr}\right)\times\left(-i\omega \frac{\cancel{\mu_0}}{4\pi}\frac{e^{ikr}}{r}\vec{p}\right)$$

$$= \frac{-i\omega}{4\pi}\left(ik - \frac{1}{r}\right)\hat{r}\times\vec{p}\,\frac{e^{ikr}}{r}.$$

$$\boxed{\vec{H}_{E.D.} = \frac{ck^2}{4\pi}\hat{r}\times\vec{p}\,\frac{e^{ikr}}{r}\left(1 - \frac{1}{ikr}\right)}$$

$$\vec{\nabla}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} = -i\omega\varepsilon_0\vec{E} = -ick\varepsilon_0\vec{E}$$

9.20

$$\boxed{\vec{E}_{E.D.} = \frac{1}{4\pi\varepsilon_0}\left[k^2(\hat{r}\times\vec{p})\times\hat{r}\,\frac{e^{ikr}}{r} + \frac{\left[3\hat{r}(\hat{r}\cdot\vec{p}) - \vec{p}\right]}{r^3}e^{ikr}(1-ikr)\right]}$$

$k \to 0$ ($kr \to 0$). $\vec{H} \to 0$, as k^2

$\vec{E} \to$ (static electric diple) $\wedge$ $e^{-i\omega t}$.

③

$kr \to \infty$. $\vec{E}, \vec{H} \sim \frac{e^{ikr}}{r}$ radiation fields.

$kr \sim 1$ intermediate, transition regime → all terms.

(9.19)

$$\langle \vec{S} \rangle = \mathrm{Re}\left(\frac{1}{2} \vec{E} \times \vec{H}^*\right)$$

$$= \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0} k^2 (\hat{r}\times\vec{p})\times\hat{r}\, \frac{e^{ikr}}{r}\right) \times \left(\frac{ck^2}{4\pi}\, \hat{r}\times\vec{p}\, \frac{e^{ikr}}{r}\right)$$

$$= \frac{1}{32\pi^2\varepsilon_0} \frac{ck^4}{r^2} \underbrace{\left[(\hat{r}\times\vec{p})\times\hat{r}\right] \times (\hat{r}\times\vec{p}^*)}_{\hat{r}\,|\hat{r}\times\vec{p}|^2}$$

radial inverse square

$$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{ck^4}{32\pi^2\varepsilon_0} \left|(\hat{r}\times\vec{p})\times\hat{r}\right|^2}$$

insert: $c\sqrt{\mu_0\varepsilon_0} = 1$, $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$

$$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c^2}{32\pi^2} Z_0 k^4 \left|(\hat{r}\times\vec{p})\times\hat{r}\right|^2} \quad (9.22)$$

$\hat{n} = \hat{r}$

$[p] = Q \cdot l$

$c^2k^2 = \omega^2$ $\Big\}$ $c^2k^4p^2 \sim \frac{Q^2}{t^2} \sim \text{current}^2$

$P \sim I^2$

$|(\hat{r}\times\vec{p})\times\hat{r}|$ contains angular dependence

$\vec{p}\propto\hat{z}$. $\quad |\hat{r}\times\hat{z}|^2 = \left|(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \cancel{\hat{z}\cos\theta})\times\hat{z}\right|^2$

$$= \left|-\hat{y}\sin\theta\cos\phi + \hat{x}\sin\theta\sin\phi\right|^2$$

donut

$$= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi = \sin^2\theta$$

$\vec{p}\sim(\hat{x}+i\hat{y})$ $\quad |\hat{r}\times(\hat{x}+i\hat{y})|^2$

$$= \Big|(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)\times\hat{x}$$

$$+\, i(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)\times\hat{y}\Big|^2$$

$$= \Big|i\hat{z}\sin\theta\overbrace{(\cos\phi + i\sin\phi)}^{e^{i\phi}} - i(\hat{x}+i\hat{y})\cos\theta\Big|^2$$

peanut

$$= \sin^2\theta + \cos^2\theta + \cos^2\theta = 1+\cos^2\theta$$

Elliptical → other possibilities

These are "basic"

(5)

$$P = \int d\Omega \, \frac{dP}{d\Omega} = \frac{c^2 Z_0 k^4}{32\pi^2} \int d\Omega \, |\hat{r} \times \vec{p}|^2$$

$$|\hat{r} \times \vec{p}|^2 = (\hat{r} \times \vec{p}) \cdot (\hat{r} \times \vec{p}^*) = \hat{r} \cdot (\vec{p} \times (\hat{r} \times \vec{p}^*))$$

$$= \hat{r} \cdot \left(\hat{r} (\vec{p} \cdot \vec{p}^*) - \vec{p}^* (\hat{r} \cdot \vec{p}) \right)$$

$$= \vec{p} \cdot \vec{p}^* - (\hat{r} \cdot \vec{p})(\hat{r} \cdot \vec{p}^*)$$

$$P = \frac{c^2 Z_0 k^4}{32\pi^2} \int d\Omega \left[\underbrace{(\vec{p} \cdot \vec{p}^*)}_{4\pi} - \underbrace{\hat{r}_i \hat{r}_j}_{\frac{4\pi}{3}\delta_{ij}} p_i p_j^* \right]$$

$$P = \frac{c^2 Z_0 k^4}{32\pi^2} |\vec{p}|^2 \underbrace{\left[4\pi - \frac{4\pi}{3} \right]}_{\frac{8\pi}{3}}$$

$$\boxed{P = \frac{c^2 Z_0}{12\pi} |\vec{p}|^2} \quad (9.24)$$

$$c Z_0 = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \sqrt{\frac{\mu_0}{\epsilon_0}} = \frac{1}{\epsilon_0} \rightarrow \frac{1}{3} \frac{c k^4 |\vec{p}|^2}{4\pi\epsilon_0}$$

equivalent

$$e^{-ik\hat{r}\cdot\hat{x}'} = 1 - \underline{\underline{ik\hat{r}\cdot\vec{x}'}} + \cdots$$

next order term.

$$(\hat{r}\cdot\vec{x}')\vec{J} = \frac{1}{2}\left[(\hat{r}\cdot\vec{x}')\vec{J} + (\hat{r}\cdot\vec{J})\vec{x}'\right] + \frac{1}{2}\left[(\hat{r}\cdot\vec{x}')\vec{J} - (\hat{r}\cdot\vec{J})\vec{x}'\right]$$

$$(\vec{x}'\times\vec{J})\times\hat{r} = \vec{J}(\hat{r}\cdot\vec{x}') - \vec{x}'(\hat{r}\cdot\vec{J})$$

Asymmetric piece.

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\,\frac{1}{2}(\vec{x}'\times\vec{J})\times\hat{r}\,(-ik).$$

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\,ik\hat{r}\times\vec{m}}$$

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{A} = \frac{ik\hat{r}}{\mu_0}\times\vec{A} = -\frac{k^2}{4\pi}\frac{e^{ikr}}{r}\,\hat{r}\times(\hat{r}\times\vec{m})$$

$$\boxed{\vec{H} = \frac{k^2}{4\pi}\frac{e^{ikr}}{r}(\hat{r}\times\vec{m})\times\hat{r}}$$

$$\vec{\nabla}\times\vec{H} = ik\hat{r}\times\vec{H} = ik\left(\frac{k^2}{4\pi}\right)\frac{e^{ikr}}{r}\hat{r}\times\vec{m} = -i\omega\epsilon_0\vec{E}$$

$$\boxed{\vec{E} = -\frac{1}{4\pi\epsilon_0}\frac{k^2}{c}\frac{e^{ikr}}{r}\hat{r}\times\vec{m}}$$

Electric/magnetic Duality

$$\vec{E} \to Z_0 \vec{H} \quad (c\vec{B})$$
$$Z_0 \vec{H} \to -\vec{E}$$

$$\vec{p} \to \frac{\vec{m}}{c}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} k^2 (\hat{r} \times \vec{p}) \times \hat{r} \frac{e^{ikr}}{r}$$

$$\to Z_0 \vec{H} = \frac{1}{4\pi\varepsilon_0} k^2 \left(\hat{r} \times \frac{\vec{m}}{c}\right) \times \hat{r} \frac{e^{ikr}}{r}$$

$$\vec{H} = \frac{k^2}{4\pi} \left(\frac{1}{Z_0 \varepsilon_0 c}\right) \left(\hat{r} \times \vec{m}\right) \times \hat{r}$$

$$\hookrightarrow \sqrt{\frac{\mu_0}{\varepsilon_0}} \varepsilon_0 \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 1.$$

$$\boxed{\left(\frac{dP}{d\Omega}\right) = \frac{1}{32\pi^2} Z_0 k^4 \left|\hat{r} \times \vec{m}\right|^2}$$

$$\boxed{\langle P \rangle = \frac{c^2 Z_0 k^4}{12\pi} \left|\frac{\vec{m}}{c}\right|^2}$$

Symmetric part.

$$\hat{A} = -ik \cdot \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \; \frac{1}{2}\left[(\hat{r}\cdot\vec{x}')\vec{J} + (\hat{r}\cdot\vec{J})\vec{x}' \right]$$

$$\hat{r}_j \left(x_i' J_j + x_j' J_i \right)$$

$$\nabla_k' \left(x_i' x_j' J_k \right) = \delta_{ik} x_j' J_k + x_i' \delta_{jk} J_k + x_i' x_j' \nabla_k' J_k$$

$$= x_i' J_j + x_j' J_i + \underline{i\omega\rho \, x_i' x_j'}$$

$$\int d^3x \, \nabla_k (\;) = 0 = \int d^3x' \left(x_i' J_j + x_j' J_i + i\omega\rho \, x_i' x_j' \right)$$

$$\Rightarrow \quad \boxed{\vec{A} = -ck^2 \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \frac{1}{2} \hat{r}_j \int d^3x' \, x_i' x_j' \rho}$$

recall: $Q_{ij} = \int d^3x' \left(3x_i' x_j' - \delta_{ij} r'^2 \right) \rho$.

gauge

$$\vec{A} = -ck^2 \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \frac{1}{6} \hat{r}_j \left(Q_{ij} + C \cdot \delta_{ij} \right)$$

$$\boxed{\hat{A} = \frac{-ck^2 \mu_0}{24\pi} \frac{e^{ikr}}{r} \left(\vec{Q}(\hat{r}) + C\hat{r} \right)}$$

9.38

$$\vec{Q}(\hat{r}) = Q_{ij} \hat{r}_j$$