2/19/2018 Radiation $(r \gg d)$.

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int d^3x'\ \vec{J}(\vec{x}')\, e^{-ik\hat{r}\cdot\vec{x}'}$$

$kd \ll 1$ $\quad e^{-ik\hat{r}\cdot\vec{x}'} \simeq 1 - ik\hat{r}\cdot\vec{x}'$

$$\vec{J}(\hat{r}\cdot\vec{x}') = \frac{1}{2}\left[\vec{J}(\hat{r}\cdot\vec{x}') + \vec{x}'(\hat{r}\cdot\vec{J})\right]$$

$$+\frac{1}{2}\left[\vec{J}(\hat{r}\cdot\vec{x}') - \vec{x}'(\hat{r}\cdot\vec{J})\right]$$

$\longleftrightarrow$ magnetic dipole moment

$$\boxed{\vec{A}_{mag} = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\, ik\hat{r}\times\vec{m}}$$

$$\vec{H} = \frac{\vec{\nabla}\times\vec{A}}{\mu_0} = \frac{ik\hat{r}}{\mu_0}\times\vec{A}$$

$$\boxed{\vec{H} = \frac{k^2}{4\pi}\frac{e^{ikr}}{r}(\hat{r}\times\vec{m})\times\hat{r}}$$

$$\vec{\nabla}\times\vec{H} = ik\hat{r}\times\vec{H} = \varepsilon_0\frac{\partial\vec{E}}{\partial t} = -i\omega\varepsilon_0\vec{E}$$

$$\boxed{\vec{E} = -\frac{1}{4\pi\varepsilon_0}\frac{k^2}{c}\frac{e^{ikr}}{r}\,\hat{r}\times\vec{m}}$$

Electric/magnetic duality

$$\vec{E} \to Z_0 \vec{H} \quad (c\vec{B})$$
$$Z_0 \vec{H} \to -\vec{E}$$

$$\vec{p} \to \frac{\vec{m}}{c}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} k^2 (\hat{r} \times \vec{p}) \times \hat{r} \frac{e^{ikr}}{r}$$

$$\to Z_0 \vec{H} = \frac{1}{4\pi\varepsilon_0} k^2 \left(\hat{r} \times \frac{\vec{m}}{c}\right) \times \hat{r} \frac{e^{ikr}}{r}$$

$$\vec{H} = \frac{k^2}{4\pi} \left(\frac{1}{Z_0 \varepsilon_0 c}\right) \left(\hat{r} \times \vec{m}\right) \times \hat{r}$$

$$\hookrightarrow \sqrt{\frac{\mu_0}{\varepsilon_0}}\, \varepsilon_0 \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 1.$$

$$\boxed{\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{32\pi^2} Z_0 k^4 \left| \hat{r} \times \vec{m} \right|^2}$$

$$\boxed{\langle P \rangle = \frac{c^2 Z_0 k^4}{12\pi} \left| \frac{\vec{m}}{c} \right|^2}$$

Symmetric part.

$$\vec{A} = -ik \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \frac{1}{2}\left[(\hat{r}\cdot\vec{x}')\vec{J} + (\hat{r}\cdot\vec{J})\vec{x}' \right]$$

$$\hat{r}_j (x'_i J_j + x'_j J_i)$$

$$\nabla'_k (x'_i x'_j J_k) = \delta_{ik} x'_j J_k + x'_i \delta_{jk} J_k + x'_i x'_j \nabla'_k J_k$$

$$= x'_i J_j + x'_j J_i + i\omega\rho \, x'_i x'_j$$

$$\int d^3x \, \nabla_k(\;) = 0 = \int d^3x' \, (x'_i J_j + x'_j J_i + i\omega\rho \, x'_i x'_j)$$

$$\Rightarrow \boxed{\vec{A} = -ck^2 \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \frac{1}{2} \hat{r}_j \int d^3x' \, x'_i x'_j \rho}$$

recall: $Q_{ij} = \int d^3x' \, (3x'_i x'_j - \delta_{ij} r'^2)\rho$.

$$\vec{A} = -ck^2 \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \frac{1}{6} \hat{r}_j (Q_{ij} + \overset{\text{gauge}}{C} \cdot \delta_{ij})$$

$$\boxed{\vec{A} = \frac{-ck^2 \mu_0}{24\pi} \frac{e^{ikr}}{r} \left(\vec{Q}(\hat{r}) + C\hat{r} \right)} \quad 9.38$$

$$\boxed{\vec{Q}(\hat{r}) = Q_{ij} \hat{r}_j}$$

$$\vec{H} = \frac{1}{\mu_0} ik\hat{r}\times\vec{A}$$

$$\vec{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\hat{r}\times\vec{Q}(\hat{r})$$

$\varepsilon_{ijk}\hat{r}_j Q_{km}\hat{r}_m$

$$\vec{E} = Z_0\vec{H}\times\hat{r}$$

$$\vec{\nabla}\times\vec{H} = ik\hat{r}\times\vec{H} = \frac{\partial\vec{D}}{\partial t} = -i\omega\varepsilon_0\vec{E}$$

$$\rightarrow \vec{E} = -\frac{k}{\omega\varepsilon_0}\hat{r}\times\vec{H} = +\frac{1}{\varepsilon_0 c}\vec{H}\times\hat{r} = Z_0\vec{H}\times\hat{r}$$

$$\frac{dP}{d\Omega} = \frac{1}{2}r^2\hat{r}\cdot\vec{E}\times\vec{H}^*$$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{\underset{\uparrow\, 2\times(24)^2}{1152\pi^2}}\left|\left(\hat{r}\times\vec{Q}(\hat{r})\right)\times\hat{r}\right|^2 \qquad (9.45)$$

$$Q_{ij} = \int d^3x' \left(3x'_i x'_j - \delta_{ij} r'^2\right)\rho.$$

rotationally symmetric $\longrightarrow$ $\underset{xy}{\sin\phi'}$, $\underset{xz}{\cos\phi'}$, $\underset{yz}{\cos\phi'\sin\phi'} \to 0$

$(xx),(yy)$ $\quad \int d^3x' \left(3r'^2\sin^2\theta\sin^2\phi' - r'^2\right)$

$$\hookrightarrow \frac{3}{2}\sin^2\theta - 1 = \frac{3}{2}(1-\cos^2\theta) - 1 = \frac{1}{2} - \frac{3}{2}\cos^2\theta$$

(zz) $\quad 3\cos^2\theta - 1 \qquad Q_{xx} = Q_{yy} = -\frac{1}{2}Q_{zz}.$

5

$$Q_{ij} = \begin{pmatrix} -\frac{1}{2}Q_0 & & \\ & -\frac{1}{2}Q_0 & \\ & & Q_0 \end{pmatrix}$$

$$\vec{Q}(\hat{r}) = Q_0\left(-\tfrac{1}{2}\sin\theta\cos\phi\,\hat{x} - \tfrac{1}{2}\sin\theta\sin\phi\,\hat{y} + \hat{z}\cos\theta\right)$$

$$= Q_0\left(-\tfrac{1}{2}\hat{r} + \tfrac{3}{2}\cos\theta\,\hat{z}\right).$$

$$= -\sin\theta\,\hat{\phi}$$

$$\hat{r}\times\vec{Q} = \tfrac{3}{2}Q_0\cos\theta\,(\hat{r}\times\hat{z}) = \sin\theta\,(+\hat{x}\sin\phi - \hat{y}\sin\phi)$$

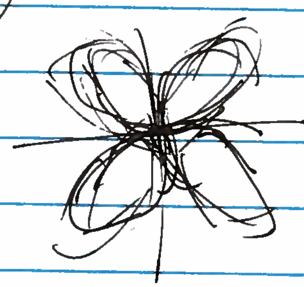
$$(\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \cancel{\hat{z}\cos\theta})\times\hat{z}$$

$$\vec{B} \propto \left(\tfrac{3}{2}Q_0\right)\sin\theta\cos\theta\,\hat{\phi}$$

$$\vec{E} \propto \vec{B}\times\hat{r} \propto \left(\tfrac{3}{2}Q_0\right)\sin\theta\cos\theta\cdot\hat{\theta}$$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0 k^6}{\underbrace{1152\pi^2}_{512\pi^2}}\cdot\frac{9}{4}|Q_0|^2\sin^2\theta\cos^2\theta$$

9.51



Total Power:

$$|(\hat{r}\times\vec{Q})\times\hat{r}|^2 = |\hat{r}\times\vec{Q}|^2$$

$$= (\hat{r}\times\vec{Q})\cdot(\hat{r}\times\vec{Q}^*) = \hat{r}\cdot[\vec{Q}\times(\hat{r}\times\vec{Q}^*)]$$

$$= \vec{Q}\cdot\vec{Q}^* - (\hat{r}\cdot\vec{Q})(\hat{r}\cdot\vec{Q}^*)$$

$$= Q_{ij}\hat{r}_j Q^*_{ik}\hat{r}_k - Q_{ij}\hat{r}_i\hat{r}_j Q_{mn}\hat{r}_m\hat{r}_n.$$

$$\int d\Omega\, \hat{r}_i\hat{r}_j = \frac{4\pi}{3}\delta_{ij}$$

$$\int d\Omega\, \hat{r}_i\hat{r}_j\hat{r}_m\hat{r}_n = K\cdot(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})$$

$\underline{i=j=m=n=3}$ $\quad \int d\Omega\, \cos^4\theta = 2\pi\int_{-1}^{1} dx\, x^4 = \frac{4\pi}{5} = K\cdot(1+1+1)$

$$\int d\Omega\, \hat{r}_i\hat{r}_j\hat{r}_m\hat{r}_n = \frac{4\pi}{15}(\delta_{ij}\delta_{mn} + \delta_{in}\delta_{jm} + \delta_{im}\delta_{jn})$$

(9.47) (dB)

$$\int d\Omega\, |\hat{r}\times\vec{Q}|^2 = Q_{ij}Q^*_{ik}\cdot\frac{4\pi}{3}\delta_{ij}$$

$$- Q_{ij}Q^*_{mn}\cdot\frac{4\pi}{15}(\cancel{\delta_{ij}\delta_{mn}} + \delta_{in}\delta_{jm} + \delta_{im}\delta_{jn})$$

$$= Q_{ij}Q^*_{ij}\left(\frac{4\pi}{3} - \frac{4\pi}{15}\cdot 2\right)$$

$$\frac{4\pi}{15}(5-2) = \frac{4\pi}{15}\cdot 3 = \frac{4\pi}{5}.$$

$$P = \frac{c^2 Z_0 k^6}{1152\pi^2} \cdot \frac{4\pi}{15} \cdot Q_{ij} Q_{ij}^*$$

$$\boxed{P = \frac{c^2 Z_0 k^6}{1440\pi} |Q_{ij}|^2} \qquad (9.49)$$

Gravity $\qquad cZ_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{\varepsilon_0}$

$$P = \frac{1}{360} \cdot ck^6 \cdot \frac{|Q_{ij}|^2}{4\pi\varepsilon_0}$$

Analogy. $\frac{q^2}{4\pi\varepsilon_0} \to Gm^2$

$$Q_{ij} = \int d^3x \, (3x_i x_j - \delta_{ij} r^2)$$

$$I_{ij} = \int d^3x \left(x_i x_j - \frac{1}{3}\delta_{ij} r^2\right)$$

$$P \to \frac{1}{360} \cdot ck^6 G (9 I_{ij})^2 = \frac{1}{40} ck^6 G |I_{ij}|^2$$

GR $\qquad \sqrt{P = \frac{1}{10} \cdot ck^6 \cdot G |I_{ij}|^2}$

$\sqrt{\text{Rotating}}$

$P = \frac{G}{5c^5} \langle \dddot{I}_{ij} \dddot{I}_{ij} \rangle$